Global stability for the 3-dimensional logistic map

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Abstract. For the delayed logistic equation $x_{n+1} = ax_n(1 - x_{n-2})$ it is well known that the nontrivial fixed point is locally stable for $1 < a \leq \left(\sqrt{5}+1\right)/2$, and unstable for $a > \left(\sqrt{5}+1\right)/2$. We prove that for $1 < a \leq \left(\sqrt{5}+1\right)/2$ the fixed point is globally stable, in the sense that it is locally stable and attracts all points of S, where S contains those $(x_0, x_1, x_2) \in \mathbb{R}^3_+$ for which the sequence $(x_n)_{n=0}^{\infty}$ remains in $\mathbb{R}_+$. The proof is a combination of analytical and reliable numerical methods. The novelty of this article is an explicit construction of a relatively large attracting neighborhood of the nontrivial fixed point of the 3-dimensional logistic map by using center manifold techniques and the Neimark–Sacker bifurcational normal form.

Keywords: Delayed logistic map; global stability; Neimark–Sacker bifurcation; center manifold; interval analysis; rigorous numerics; graph representation

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1 Introduction

One of the most studied nonlinear maps is the logistic map $[0, 1] \ni x \mapsto ax(1 - x) \in \mathbb{R}$ with parameter $a > 0$. It is well known (see e.g., [1]) that $x = 0$ is the unique fixed point in $[0, 1]$ for $0 < a \leq 1$, and it is globally stable in $[0, 1]$. For $1 < a \leq 3$ the nontrivial fixed point $x^* = 1 - 1/a$ is stable and attracts all points of $(0, 1)$. At $a = 3$ a period doubling bifurcation takes place, and the fixed point x^* becomes unstable for $a > 3$. As a increases, there is a sequence of bifurcations, and for some larger value of a, chaotic behavior can be shown.

In 1971 Levin and May [2] considered the delayed logistic difference equation

$$x_{n+1} = ax_n(1 - x_{n-d})$$

with $a > 0$ and $d \in \mathbb{N}$. This is natural in the context of population models; the size of the subsequent generation of the population depends not only on the size in the previous year, but also on the size of the d-year-earlier population.

In case $d = 1$, it is easy to see that there is a nontrivial positive fixed point for $a > 1$, which is locally asymptotically stable for $a \in (1, 2)$, and unstable for $a > 2$. In [3] it was shown that this nontrivial fixed point is also globally stable for $a \in (1, 2]$ in the sense that it is locally stable and attracts every point of the set $\{(u_1, u_2) \in \mathbb{R}^2 : \ u_1 \in [0, 1), \ \ u_2 \in (0, 1), \ \ au_2(1 - u_1) < 1\}$.

Bartha, Garab and Krisztin in [4, 5] proved analogous results for other second order difference equations, or equivalently, for other 2-dimensional maps. The novelty of [3, 4, 5] is the development of a new method to show sharp results for global stability of fixed points for some 2-dimensional maps with a parameter a. It is common in [3, 4, 5] that a supercritical Neimark–Sacker bifurcation takes place at some $a = a_{crit}$. The Neimark–Sacker bifurcational normal form not only guarantees the existence of an invariant curve around the fixed point for

$a > a_{crit}$, but also for $a \leq a_{crit}$ it gives a neighborhood $\mathcal{M}$ around the fixed point so that $\mathcal{M}$ belongs to the region of attraction of the fixed point. The main achievement of [3, 4, 5] is an explicitly constructed $\mathcal{M}$ which is large enough in the sense that, by using a rigorous computer-assisted technique, it is possible to prove that the iterates of all points outside $\mathcal{M}$ eventually enter $\mathcal{M}$. A relatively large $\mathcal{M}$ can guarantee the success of the computer-assisted part within a reasonable computer time. It is a highly nontrivial result of [3, 4, 5] that starting from the classical Neimark–Sacker bifurcational normal form technique, which was used earlier only for local results, a relatively large attractivity region can be explicitly constructed for $a \leq a_{crit}$. In [3, 4, 5] it was essential that the studied systems were 2-dimensional.

The primary aim of this paper is to extend the method of [3, 4, 5] from 2-dimensional to higher-dimensional maps. As the delayed difference equation is interesting in its own right for $d = 2$, we study the difference equation

$$x_{n+1} = ax_n(1 - x_{n-2}), \tag{1}$$

which is equivalent to the 3-dimensional map

$$F_a : \mathbb{R}^3 \ni u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ u_3 \\ au_3(1 - u_1) \end{pmatrix} \in \mathbb{R}^3. \tag{2}$$

On this 3-dimensional nonlinear map we demonstrate how the extension goes to higher dimension. We believe that the main steps of our case study for (2) can be followed with natural modifications to handle several other problems, in particular the delayed logistic map for $d > 2$.

Define $a_0 = \left(\sqrt{5} + 1\right)/2$, and consider the map F_a for those $u \in \mathbb{R}^3_+ = [0, \infty)^3$ for which all iterates of (2) remain in $\mathbb{R}^3_+$, i.e., $F_a^n(u) \in \mathbb{R}^3_+$, for every $n \in \mathbb{N}$. Here, F_a^n denotes the n-fold iteration of F_a, i.e., $F_a^0 = \mathrm{id}$ and $F_a^n = F_a\left(F_a^{n-1}\right)$, $n \in \mathbb{N}$. As we will see for $a \in (0, a_0]$, the set

$$\tilde{S} = \tilde{S}(a) = \left\{u \in \mathbb{R}^3 : \ u_1, u_2, u_3 \in [0, 1], \quad au_3(1 - u_1) \leq 1, \quad a^2u_3(1 - u_1)(1 - u_2) \leq 1\right\}$$

is the largest set, whose points remain in $\mathbb{R}^3_+$ for all iterates F_a^n, $n \in \mathbb{N}$. Note that $\tilde{S}$ depends on the parameter a.

For $a \in (0, 1]$ we have $\tilde{S} = [0, 1]^3$, and the only fixed point of (2) in $\tilde{S}$ is the origin. It is elementary to show that the origin is locally stable and $\lim_{n\to\infty} F_a^n(u) = 0$ for every $u \in \tilde{S}$. For $a > 1$ a nontrivial fixed point $u_A = (A, A, A)$ with $A = 1 - 1/a$ appears in $\tilde{S}$. This fixed point is locally asymptotically stable for $a \in (1, a_0)$, and unstable for $a > a_0$. A Neimark–Sacker bifurcation takes place at $a = a_0$, and there is a stable invariant curve for $a > a_0$ sufficiently close to a_0.

In this paper we show the globally stability of the nontrivial fixed point u_A for $a \in (1, a_0]$. Our main result is the following theorem.

Theorem 1. *The fixed point u_A is locally stable, and $\lim\limits_{n\to\infty} F_a^n(u) = u_A$ for every $a \in (1, a_0]$ and $u \in S(a)$, where*

$$S(a) = \left\{u \in \mathbb{R}^3 : \ u_1, u_2 \in [0, 1), \quad u_3 \in (0, 1), \quad au_3(1 - u_1) < 1, \quad a^2u_3(1 - u_1)(1 - u_2) < 1\right\}.$$

We emphasize that Theorem 1 is sharp, that is, global stability holds at the critical parameter value a_0 as well. Theorem 1 can be formulated so that local stability implies global stability for the fixed point u_A. This is satisfied for several problems, see e.g., [6, 7], but it is not true in general (see e.g., [8]). Note that we do not consider the case $a > a_0$. However, local information is available from the Neimark–Sacker bifurcation near u_A for $a > a_0$ close to a_0. Analogously to the cases $d = 0$ and $d = 1$, it is expected that the dynamics becomes more complex with larger a.

In Section 2 we describe the behavior of (2) in the positive octant. Then, for $a \in (1, 4/3]$ we give a purely analytical proof of the global stability of u_A. Although this analytical proof is straightforward, it is important toward the proof of Theorem 1, since for $a > 1$ and close to 1 the fixed point u_A can not be distinguished from the origin by a computer-assisted technique.

The rest of the article is devoted to the case $a \in (4/3, a_0]$. The basic idea is the same as for the 2-dimensional case. First, we analytically construct an attracting neighborhood $\mathcal{M}(a)$ of the fixed point u_A. Then, by applying reliable numerical tools, it is shown that for every $u \in S$ the iterates $F_a^n(u)$ eventually enter $\mathcal{M}(a)$. Consequently, all points of S belong to the region of attraction of u_A. Here, reliable means that all possible numerical errors are controlled by using interval arithmetic techniques. Therefore, the computer-assisted part also provides mathematically rigorous statements.

In Section 3 the map F_a is transformed to the form

$$H_a :\ \mathbb{C} \times \mathbb{R} \ni \begin{pmatrix} z \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda(a)z + \mathcal{G}_a(z, y) \\ \nu(a)y + \mathcal{H}_a(z, y) \end{pmatrix} \in \mathbb{C} \times \mathbb{R} \tag{3}$$

for $a \in (4/3, a_0]$. Here, $\nu(a) \in \mathbb{R}$ and $\lambda(a) \in \mathbb{C}$ with $\bar{\lambda}(a) \in \mathbb{C}$ are the eigenvalues of the Jacobian matrix of (2) at u_A. The inequality $|\nu(a)| < |\lambda(a)| \leq 1$ holds and $|\lambda(a_0)| = 1$. The nonlinear functions $\mathcal{G}_a(z, y)$ and $\mathcal{H}_a(z, y)$ are smooth functions of $a, z, \bar{z}$ and y, and furthermore, they are $O\left(|(z, y)|^2\right)$ for each fixed $a \in (4/3, a_0]$.

In Section 3.1 a standard linearization technique for (3) gives an attracting neighborhood of u_A for $a \in (4/3, a_0)$. When $a \to a_0^-$, the attracting neighborhood obtained via linearization shrinks to the fixed point. Therefore, a different approach is necessary for parameter values close to a_0. In the subsequent sections, for $a \in \mathcal{I}_0 = [a_0 - 10^{-2}, a_0]$ we adapt the technique from [3, 4] based on the Neimark–Sacker bifurcational normal form. However, we need new ideas, since (2) is 3-dimensional, and thus the adaptation of the method from [3] is not that straightforward.

The classical way to study the dynamics of the 3-dimensional map F_a near u_A, or equivalently, the dynamics of H_a near $(0, 0) \in \mathbb{C} \times \mathbb{R}$, for a close to a_0 is as follows. First, a center manifold reduction is carried out, then the map is transformed to its normal form on the center manifold, and finally the attraction property of the center manifold is used. These steps together give a local information on the dynamics of (3) for a is close to a_0 and (z, y) is close to $(0, 0)$. In particular, for $a \leq a_0$ and a close to a_0, local stability is obtained for the fixed point of (3). The major achievement of this paper is the elaboration of a quantitative version of the above local bifurcation result so that an attractive neighborhood of the fixed point $(0, 0)$ can be explicitly constructed. Sections 4–6 are devoted to this issue. In Sections 7 and 8 it turns out that the constructed attracting neighborhood is large enough, and we can handle the remaining points by a rigorous numerical technique.

In Section 4 we consider an approximated version of the center manifold reduction. It is well known that for each fixed a there is a local invariant manifold $\mathcal{W}_a^c$ of map (3) at $(0, 0)$ given by the graph

$$\mathcal{W}_a^c = \{(z, y) \in \mathbb{C} \times \mathbb{R} :\ y = \Phi_a(z),\ |z| < \delta\}$$

of a smooth map $\Phi_a : \{z \in \mathbb{C} :\ |z| < \delta\} \to \mathbb{R}$ with some $\delta = \delta(a) > 0$ and $\Phi_a(0) = 0$, $\Phi_a(z) = O(|z|^2)$. Note that $\Phi_a(\cdot)$ is not complex differentiable, but it is a smooth function of z and $\bar{z}$. The set $\mathcal{W}_a^c$ is a so called generalized center (or center-unstable) manifold of (3) at $(0, 0)$ corresponding to the leading eigenvalues $\lambda(a), \bar{\lambda}(a)$ (see [9]). The invariance property of $\mathcal{W}_a^c$ means that $H_a(z, \Phi_a(z)) \in \mathcal{W}_a^c$ for $z \in \mathbb{C}$ with small $|z|$. Or equivalently, $\mathcal{N}(\Phi_a(z)) = 0$, where $\mathcal{N}$ is given by

$$\mathcal{N}(\mathcal{F}(z)) = \mathcal{F}\Big(\lambda(a)z + \mathcal{G}_a\big(z, \mathcal{F}(z)\big)\Big) - \nu(a)\mathcal{F}(z) - \mathcal{H}_a\big(z, \mathcal{F}(z)\big),$$

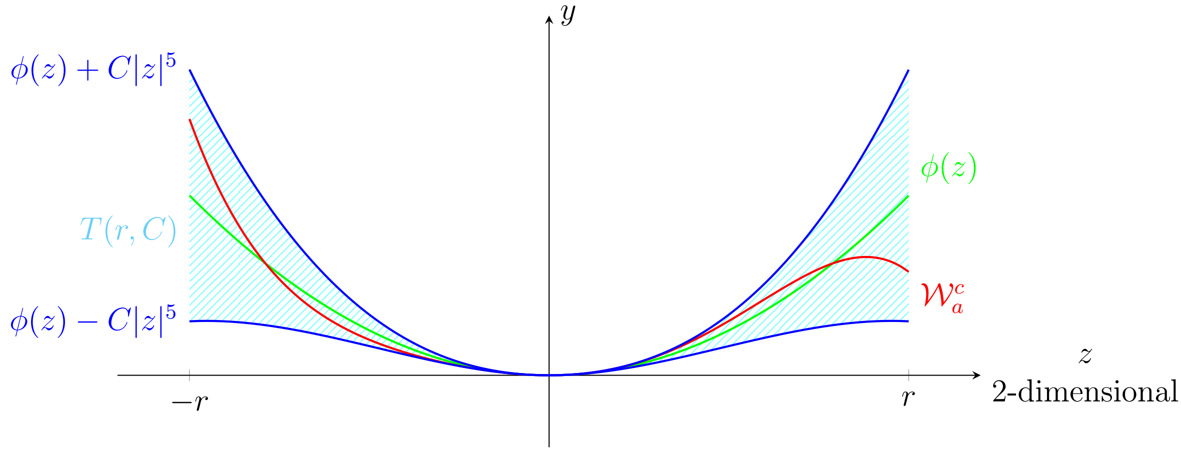


Figure 1. The set $T(r, C)$ around the approximation of the center manifold

for all maps $\mathcal{F} : \mathbb{C} \to \mathbb{R}$.

The map $\Phi_a(z)$ is problematic concerning its use in quantitative estimations for several reasons. First, it is obtained from a global manifold via modification of the nonlinearity, and its domain can be too small for computational purposes. Furthermore, it is not unique, and there is no explicit formula for it, either. In addition, we need a 3-dimensional attracting set around the fixed point for the computer-aided part, and thus a 2-dimensional set on the manifold is not sufficient. Therefore, we consider a polynomial approximation of $\Phi_a(z)$, instead. Namely, for each $a \in \mathcal{I}_0$ there is a unique fourth order (in $z, \bar{z}$) polynomial

$$\phi(z) = \phi_a(z, \bar{z}) = \sum_{n=2}^{4} \sum_{i+j=n} \frac{1}{i!j!} \omega_{ij}(a) z^i \bar{z}^j$$

with

$$\mathcal{N}(\phi(z)) = O(|z|^5).$$

The advantage of $\phi(z)$ comparing to $\Phi_a(z)$ is that it is defined on the whole complex plane, it is unique, and the coefficients $\omega_{ij}(a)$ can be easily determined and estimated. The disadvantage is that the graph of $\phi(z)$ is not locally invariant under H_a any more. However, a particular 3-dimensional set T containing the graph of $y = \phi(z)$ behaves similarly to a center manifold, and in our work it takes over the role of $\mathcal{W}_a^c$. It will turn out that T is inside the region of attraction of the fixed point.

Define the set

$$T(r, C) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \le r, \ |y - \phi(z)| \le C|z|^5\}$$

around $y = \phi(z)$ (see Figure 1), where r and C are some positive constants. Note that $T(r, C)$ has a relatively simple form for computational purposes. The term $C|z|^5$ in the definition of $T(r, C)$ guarantees that $T(r, C)$ contains the invariant manifold $\mathcal{W}_a^c$ for small $|z|$. The reason for this special shape of $T(r, C)$, more precisely the term $C|z|^5$, is that the normal form technique needs to be applicable for every $(z, y) \in T(r, C)$.

In Subsections 4.1 and 4.2 we investigate the y-directional dynamics. We use the property that solutions close to the fixed point decay exponentially to $T(r, C)$ since $|\nu(a)| < |\lambda(a)| \le 1$. From this it can be shown that $T(r, C)$ is conditionally invariant in direction y for a fixed r, provided that C is sufficiently large. More precisely, for a fixed r and C we show that $H_a(T(r, C)) \subseteq T(\tilde{r}, C)$ with some $\tilde{r} \ge r$. This means that for $(z_0, y_0) \in T(r, C)$ and $(z_1, y_1) =$

$H_a(z_0, y_0) \notin T(r, C)$ we must have $|z_1| > r$, that is, the image under H_a can leave $T(r, C)$ only in direction z.

In Section 5 the z-directional dynamics is investigated by using the Neimark–Sacker bifurcational normal form technique from [3]. For every $(z_0, y_0) \in T(r, C)$ the y-coordinate can be written in the form $y_0 = \phi(z_0) + c|z_0|^5$ for some $c \in \mathbb{R}$ with $|c| \le C$. Thus, for $(z_1, y_1) = H_a(z_0, y_0)$ the z-coordinate is determined by $z_1 = G(z_0)$, where

$$G(z) = G_{a,c}(z, \bar{z}) = \lambda z + \mathcal{G}_a\big(z, \phi(z) + c|z|^5\big). \tag{4}$$

For a fixed $a \in \mathcal{I}_0$ and $c \in \mathbb{R}$ with $|c| \le C$ we can transform (4) into a normal form. Namely, a nonlinear invertible map $h : \mathbb{C} \to \mathbb{C}$ can be given such that

$$w \mapsto h^{-1}(G(h(w))) = \lambda w + c_1 w^2 \overline{w} + R_2(w, \overline{w}, a, c), \tag{5}$$

where $c_1 = c_1(a, c)$ is the Lyapunov-coefficient and $R_2(w, \overline{w}, a, c) = O\left(|w|^4\right)$. It is important (see [3]) that the transformation h is completely determined by the lower order terms of (4). Because of the special shape of $T(r, C)$, parameter c appears only in the higher order terms of G, i.e., only in $R_2(w, \overline{w}, a, c)$. Consequently, h is independent of c, and $w = h(z)$ can be considered as a coordinate transformation of the whole set $T(r, C)$.

Applying the normal form method from [3], we obtain

$$\big|\lambda w + c_1 w^2 \overline{w} + R_2\big| < |w|$$

for every sufficiently small $w \neq 0$. This means that map (5) is a contraction. Consequently, H_a is a contraction in the new w-coordinate. Hence, combining the y- and the z-directional dynamics in Section 5.7, we obtain that $T(\hat{r}, C)$, with some $\hat{r} < r$, is in the region of attraction of the fixed point $(0, 0)$ of (3).

However, $T(\hat{r}, C)$ is clearly not a proper neighborhood of the origin in $\mathbb{C} \times \mathbb{R}$. Therefore, in Section 6 we define the set

$$\tilde{T}(r, K) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \le r, \ |\phi(z) - y| \le K\}$$

for some $r > 0$ and $K > 0$. By using the exponential y-directional attractivity of $T(r, C)$ we show that $\tilde{T}(r, K)$ is in the region of attraction of the fixed point. The neighborhood $\tilde{T}$ of the fixed point is suitable for the computer-assisted part of the proof.

In Sections 7 and 8 we describe the computer-assisted part of our method. We cover S with finitely many small cubes. Considering these cubes as vertices of a graph we introduce a directed graph, which, to a certain extent, describes the behavior of map (2) on these cubes. Therefore, we convert the issue of examining infinitely many points into a finite graph problem, which can be handled by computer. To construct the edges of this graph we use reliable numerical methods in order to handle the rounding errors of the computer. We show with the help of this graph that the iterates of every point from $S(a)$ enter the neighborhood constructed before, and the proof of Theorem 1 is completed.

Despite the fact that we demonstrate our method only on a specific equation, we believe that it can be applied or extended to other similar maps. For instance the Ricker map (see [4]) and the Pielou map (see [10]) with delay $d = 2$ essentially differs only in that they are not polynomial maps. Hence, only a slight modification would be necessary in the estimations. However, the main question is whether the obtained neighborhood is large enough for the computer-aided part of the method. These two maps along with the logistic map would also be interesting for larger delay, i.e., $d > 2$. We believe that the analytical part could be extended using only natural modifications. However, the computer-aided part can be critical in these cases, since the increasing dimension causes exponentially growing graph.

It also would be interesting to prove the existence of the unique invariant closed curve around the nontrivial fixed point for parameter values larger than the critical value. However, this question is substantially different from the one studied in this article.

2 Preliminaries

Throughout the article $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}_+$ denote the positive integers, the nonnegative integers and the nonnegative real numbers, respectively. We use also the big O notation in the sense that $f(x) = O(g(x))$ means that there exist positive numbers δ and M such that $|f(x)| \le Mg(x)$ for $|x| < \delta$.

For symbolic computation we use Wolfram Mathematica v. 11, and for reliable numerical estimation we use interval arithmetic tools of IntLab v. 9 in Matlab 2018.

In this section we study the dynamics of map (2) in the positive octant for $a > 0$. Introduce the following disjoint sets depending on a (see Figure 2).

$$
\begin{aligned}
S &= \big\{u \in \mathbb{R}^3 : \ u_1, u_2 \in [0,1), \ \ u_3 \in (0,1), \ \ au_3(1-u_1) < 1, \ \ a^2u_3(1-u_1)(1-u_2) < 1\big\} \\
\tilde{S}_0 &= \{(u_1,u_2,0) : \ u_1,u_2 \ge 0\} \\
&\quad \cup \{(1,u_2,u_3) : \ u_2 \ge 0, \ \ u_3 > 0\} \\
&\quad \cup \{(u_1,1,u_3) : \ u_1 \in [0,1), \ \ u_3 > 0\} \\
&\quad \cup \{(u_1,u_2,1) : \ u_1,u_2 \in [0,1)\} \\
&\quad \cup \{(u_1,u_2,u_3) : \ u_1,u_2 \in [0,1), \ \ u_3 \in (0,1), \ \ au_3(1-u_1) = 1\} \\
&\quad \cup \big\{(u_1,u_2,u_3) : \ u_1,u_2 \in [0,1), \ \ u_3 \in (0,1), \ \ au_3(1-u_1) < 1, \ \ a^2u_3(1-u_1)(1-u_2) = 1\big\} \\
\tilde{S}_1 &= \big\{(u_1,u_2,u_3) : \ u_1,u_2 \in [0,1), \ \ u_3 \in (0,1), \ \ au_3(1-u_1) < 1, \ \ a^2u_3(1-u_1)(1-u_2) > 1\big\} \\
\tilde{S}_2 &= \{(u_1,u_2,u_3) : \ u_1,u_2 \in [0,1), \ \ u_3 \in (0,1), \ \ au_3(1-u_1) > 1\} \\
\tilde{S}_3 &= \{(u_1,u_2,u_3) : \ u_1,u_2 \in [0,1), \ \ u_3 > 1\} \\
\tilde{S}_4 &= \{(u_1,u_2,u_3) : \ u_1 \in [0,1), \ \ u_2 > 1, \ \ u_3 > 0\} \\
\tilde{S}_5 &= \{(u_1,u_2,u_3) : \ u_1 > 1, \ \ u_2 \ge 0, \ \ u_3 > 0\}
\end{aligned}
$$

Clearly, $\mathbb{R}^3_+ = S \cup \tilde{S}_0 \cup \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3 \cup \tilde{S}_4 \cup \tilde{S}_5$. Furthermore, $S = [0,1)^2 \times (0,1)$ and $\tilde{S}_1 = \tilde{S}_2 = \emptyset$ for $0 < a \le 1$. Introduce the notation $\hat{u} = F_a(u)$ with $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$.

Proposition 2. *For all $a > 0$ and $i \in \{1,2,3,4\}$ we have*

$$F_a^7(\tilde{S}_0) = \{(0,0,0)\}, \quad F_a(\tilde{S}_5) \cap \mathbb{R}^3_+ = \emptyset \quad \text{and} \quad F_a(\tilde{S}_i) \subseteq \tilde{S}_{i+1}.$$

Furthermore, $F_a(S) \subseteq S$ for $a \in (1, a_0]$.

Proof. From the definition of F_a it is obvious that $F_a^7(\tilde{S}_0) = \{(0,0,0)\}$. It is also straightforward to check the relations $F_a(\tilde{S}_5) \cap \mathbb{R}^3_+ = \emptyset$ and $F_a(\tilde{S}_i) \subseteq \tilde{S}_{i+1}$ for $i \in \{1,2,3,4\}$.

If $u \in S$ then $\hat{u} = F_a(u)$ satisfies

$$\hat{u}_1 = u_2 \in [0,1), \ \hat{u}_2 = u_3 \in (0,1), \ \hat{u}_3 = au_3(1-u_1) \in (0,1)$$

and

$$a\hat{u}_3(1-\hat{u}_1) = a^2u_3(1-u_1)(1-u_2) < 1. \tag{6}$$

For the last inequality in the definition of S we have

$$a^2\hat{u}_3(1-\hat{u}_1)(1-\hat{u}_2) = a^3u_3(1-u_1)(1-u_2)(1-u_3).$$

For $u_3 \ge A = 1 - 1/a$ it follows from (6) that

$$a^2\hat{u}_3(1-\hat{u}_1)(1-\hat{u}_2) = a\left(a^2u_3(1-u_1)(1-u_2)\right)(1-u_3) < a(1-u_3) \le 1.$$

For $u_3 < A$, provided that $a \in (1, a_0]$, we obtain

$$a^2\hat{u}_3(1-\hat{u}_1)(1-\hat{u}_2) \le a^3u_3(1-u_3) < a^3A(1-A) = a^2 - a \le 1,$$

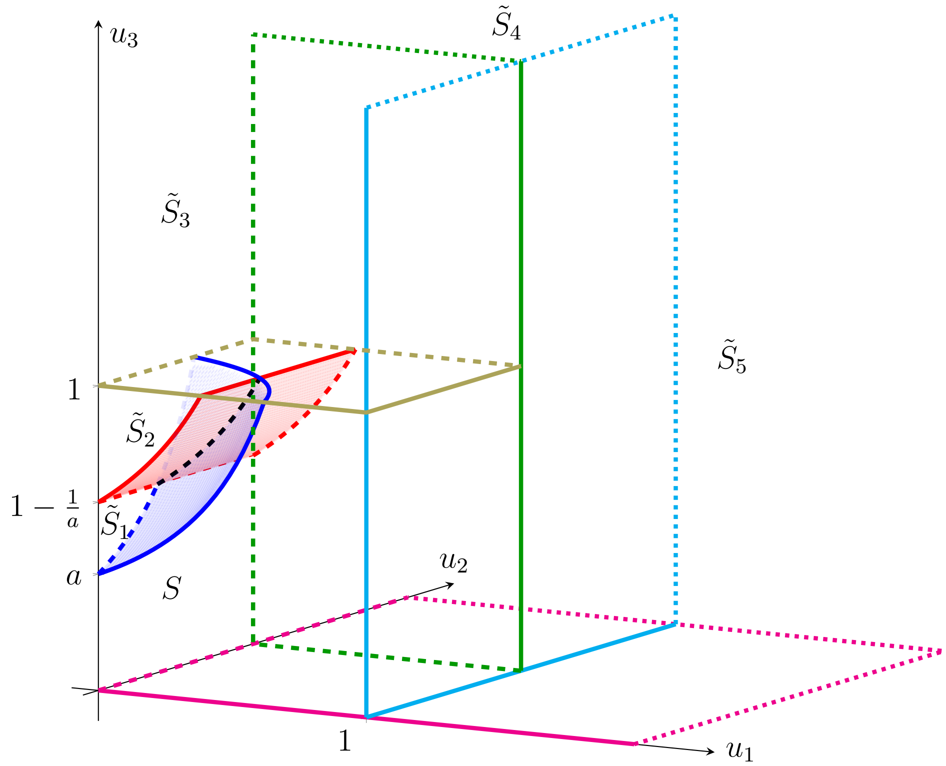


Figure 2. The subdivision of the positive octant

since $A \leq 1/2$. Thus, $F_a(S) \subseteq S$. $\square$

Consequently, in the rest of the paper we can assume that $u \in S$. For small a the dynamics in S is simple. The following statement easily follows from the fact that $x_{n+1} = ax_n(1-x_{n-2}) < x_n$, provided that $x_{n-2}, x_n \in (0,1]$ and $0 < a \leq 1$.

Proposition 3. *For every* $0 < a \leq 1$ *and* $u \in [0,1]^3$ *the following holds*

$$\lim_{n\to\infty} F_a^n(u) = (0,0,0).$$

For $a \in (1, a_0]$ we divide S into eight subsets with planes $u_1 = A$, $u_2 = A$, $u_3 = A$, and introduce the following sets.

$$\begin{aligned}
S_1 &= \{u \in S: \ u_1 \leq A, \quad u_2 \leq A, \quad u_3 < A\}\\
S_2 &= \{u \in S: \ u_1 < A, \quad u_2 > A, \quad u_3 < A\}\\
S_3 &= \{u \in S: \ u_1 \geq A, \quad u_2 > A, \quad u_3 \leq A\}\\
S_4 &= \{u \in S: \ u_1 > A, \quad u_2 \leq A, \quad u_3 \leq A\}\\
S_5 &= \{u \in S: \ u_1 \leq A, \quad u_2 < A, \quad u_3 \geq A\}\\
S_6 &= \{u \in S: \ u_1 < A, \quad u_2 \geq A, \quad u_3 \geq A\}\\
S_7 &= \{u \in S: \ u_1 \geq A, \quad u_2 \geq A, \quad u_3 > A\}\\
S_8 &= \{u \in S: \ u_1 > A, \quad u_2 < A, \quad u_3 > A\}
\end{aligned}$$

Clearly, $S = \bigcup_{i=1}^{8} S_i \cup \{u_A\}$. For given x_0, x_1, x_2 the sequence $(x_n)_{n=0}^{\infty}$, where x_n is defined by (1) for $n > 2$, corresponds to the 3-dimensional sequence $(u^n)_{n=0}^{\infty}$ with

$$u^0 = (x_0, x_1, x_2), \quad u^n = F_a^n(u^0) = (x_n, x_{n+1}, x_{n+2}), \quad n \in \mathbb{N}.$$

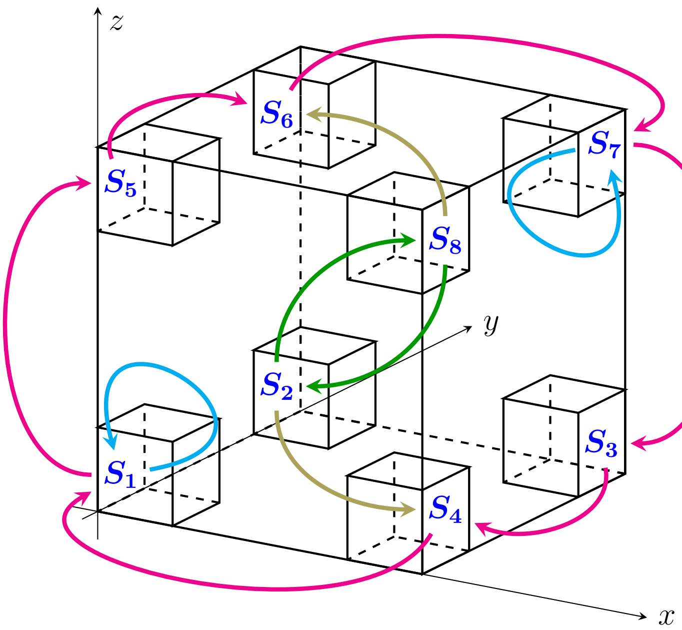
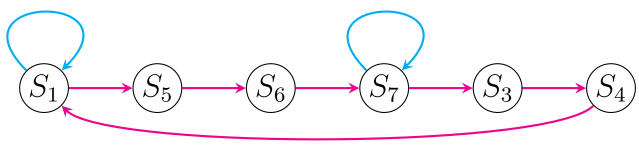


Figure 3. The dynamics in $S \setminus u_A$. The sets S_i are symbolized by a smaller cube, for the sake of transparency.

Proposition 4. *Let $a \in (1, a_0]$ and the sequence $(u^n)_{n=0}^{\infty}$ in S be given by $u^0 \neq u_A$ and $u^n = F_a^n(u^0)$, $n \in \mathbb{N}$.*

(i) The sequence $(u^n)_{n=0}^{\infty}$ follows the transition graph given in Figure 3.

(ii) If $(u^n)_{n=0}^{\infty}$ does not converge to u_A then there exists an $n_0 \in \mathbb{N}$ such that $u^{n_0} \in S_1$, $(u^n)_{n=n_0}^{\infty}$ follows the transition graph



and $(u^n)_{n=n_0}^{\infty}$ does not eventually stay in S_1 or S_7.

Proof. In order to show (i), observe the following transitions, see Figure 3.

- For $u \in S_1$ we obtain $\hat{u} = F_a(u)$ with $\hat{u}_1 \leq A$ and $\hat{u}_2 < A$. Therefore, $S_1 \rightarrowtail \{S_1, S_5\}$, that is, $\hat{u} \in S_1$ or $\hat{u} \in S_5$.
- For $u \in S_2$ we have $\hat{u}_1 > A$ and $\hat{u}_2 < A$. Thus, $S_2 \rightarrowtail \{S_4, S_8\}$.
- For $u \in S_3$ we get $\hat{u}_1 > A$, $\hat{u}_2 \leq A$ and $\hat{u}_3 = au_3(1-u_1) \leq aA(1-A) = A$, so $S_3 \rightarrowtail S_4$.
- For $u \in S_4$ we have $\hat{u}_1, \hat{u}_2 \leq A$ and $\hat{u}_3 < A$. Consequently, $S_4 \rightarrowtail S_1$.
- For $u \in S_5$ we obtain $\hat{u}_1 < A$, $\hat{u}_2 \geq A$ and $\hat{u}_3 \geq A$. Hence, $S_5 \rightarrowtail S_6$.
- For $u \in S_6$ we get $\hat{u}_1, \hat{u}_2 \geq A$ and $\hat{u}_3 > A$. Consequently, $S_6 \rightarrowtail S_7$.
- For $u \in S_7$ we have $\hat{u}_1 \geq A$ and $\hat{u}_2 > A$. Therefore, $S_7 \rightarrowtail \{S_3, S_7\}$.

- For $u \in S_8$ we have $\hat{u}_1 < A$ and $\hat{u}_2 > A$, so $S_8 \rightarrowtail \{S_2, S_6\}$.

We obtain that there is a cycle $\mathcal{C}_1 = (S_1 \rightarrowtail S_5 \rightarrowtail S_6 \rightarrowtail S_7 \rightarrowtail S_3 \rightarrowtail S_4 \rightarrowtail S_1)$. If a sequence $(u^n)_{n=0}^{\infty}$ enters the cycle $\mathcal{C}_1$ then the sequence never leaves $\mathcal{C}_1$. However, during one cycle along $\mathcal{C}_1$, the points of $(u^n)_{n=0}^{\infty}$ can spend more time in S_1 or S_7. Possibly, the sequence can get stuck and stay forever in S_1 or S_7. Furthermore, we have another cycle $\mathcal{C}_2 = (S_2 \rightarrowtail S_8 \rightarrowtail S_2)$. If a sequence steps out of the cycle $\mathcal{C}_2$ then it enters $\mathcal{C}_1$ and never returns to $\mathcal{C}_2$. Consequently, we only need to show that if a sequence $(u^n)_{n=0}^{\infty}$ gets stuck in S_1 or S_7, or in the cycle $\mathcal{C}_2$ then it converges to the fixed point u_A.

First, consider the case, when the sequence $(u^n)_{n=0}^{\infty}$ stays in $\mathcal{C}_2$. We can assume $x_0 < A$. We obtain a sequence $(x_n)_{n=0}^{\infty}$ such that $x_{2k} < A$ and $x_{2k+1} > A$, where $k \in \mathbb{N}_0$. Since

$$A < x_{2k+1} = a x_{2k}(1 - x_{2k-2}) = \frac{1}{1-A} x_{2k}(1 - x_{2k-2}),$$

we get

$$A(1-A) < x_{2k}(1 - x_{2k-2}). \tag{7}$$

Introduce the function $s(x) = x(1-x)$. Since $s(x)$ is increasing on $[0, 1/2]$ and $0 \le x_{2k-2} < A < 1/2$ we obtain $s(x_{2k-2}) < s(A)$. Combining it with (7) we get $x_{2k-2} < x_{2k}$. Consequently, $(x_{2k})_{k=0}^{\infty}$ converges monotonically to some $B \le A$. Taking the limit of both sides in inequality (7), we obtain $A(1-A) \le B(1-B)$. On the other hand $s(x)$ is increasing on $[0, A]$, so $B(1-B) \le A(1-A)$. Thereby $B = A$. The odd indexed subsequence also converges to A, since $x_{2k+1} = a x_{2k}(1 - x_{2k-2}) \to aA(1-A) = A$. So in this case, the sequence $(u^n)_{n=0}^{\infty}$ converges to the nontrivial fixed point u_A.

Now, consider the case, when the sequence gets stuck in S_1, i.e., there exists an $n_0 \in \mathbb{N}_0$ such that $(u^n)_{n=n_0}^{\infty}$ is in S_1. Notice that $u^n \in S_1$ implies $x_{n+2} \le x_{n+3}$, since $x_n \le A$. Consequently, we gain a monotone, bounded sequence $(x_n)_{n=n_0+2}^{\infty}$ which converges to some $B \le A$. Taking the limit of both sides in (1), we obtain $B = A$. Consequently, the point u^0 is in the region of attraction of u_A in this case. Similarly, if $(u^n)_{n=0}^{\infty}$ gets stuck in S_7, it also converges to the fixed point u_A. □

Now, we assume $1 < a \le 4/3$, and show that for every $u^0 \in S$ the sequence $(u^n)_{n=0}^{\infty}$ converges to the nontrivial fixed point u_A. Combining this fact with the local asymptotic stability of the fixed point (see Section 3), Theorem 1 is proven for these parameter values.

Proposition 5. *If $a \in (1, 4/3]$ and $u \in S$, then $\lim_{n\to\infty} F_a^n(u) = u_A$.*

Proof. It follows from Proposition 4 that we only need to consider the case when the sequence $(u^n)_{n=0}^{\infty}$ goes around the fixed point along the cycle $\mathcal{C}_1$, not getting stuck in S_1 or S_7. By the definition of S and $F_a(S) \subseteq S$ we obtain $x_n > 0$ for all $n \ge 2$. Without loss of generality we can assume that $u^0 \in S_1$, $x_0 > 0$ and $x_1 > 0$. Then $x_n > 0$ for all $n \in \mathbb{N}_0$. There is a strictly increasing subsequence $(n_l)_{l=0}^{\infty}$ of $\mathbb{N}_0$ such that $n_0 = 0$, $u^{n_{2k}} \in S_1$ and $u^{n_{2k+1}} \in S_7$ with

$$\begin{aligned} u^j &\notin S_7 \quad \text{for} \quad j \in \{n_{2k}, \dots, n_{2k+1} - 1\}, \\ u^j &\notin S_1 \quad \text{for} \quad j \in \{n_{2k+1}, \dots, n_{2k+2} - 1\}. \end{aligned}$$

Note that $n_{l+1} - n_l \ge 3$ holds (see the transition graph in Figure 3). Furthermore, the definition of $(n_l)_{l=0}^{\infty}$ implies

$$\begin{aligned} x_j &\le A \quad \text{for} \quad j \in \{n_{2k}, \dots, n_{2k+1} - 1\}, \\ x_j &\ge A \quad \text{for} \quad j \in \{n_{2k+1}, \dots, n_{2k+2} - 1\}. \end{aligned} \tag{8}$$

From (1) and $x_n > 0$, $n \in \mathbb{N}_0$ it is clear that $x_{n-2} \le A$ if and only if $x_n \le x_{n+1}$. Combining it with (8) we obtain $(x_j)_{j=n_{2k}+2}^{n_{2k+1}+2}$ is nondecreasing and $(x_j)_{j=n_{2k+1}+2}^{n_{2k+2}+2}$ is nonincreasing.

For the function $t: [0,1] \ni x \mapsto a^2(a-1)(1-x)^3 \in \mathbb{R}$ we have $t(A) = A$ and

$$\begin{aligned}
\frac{d}{dx}t(x) &= -3a^2(a-1)(1-x)^2 \le 0,\\
\frac{d^2}{dx^2}t(x) &= 6a^2(a-1)(1-x) \ge 0,\\
\frac{d}{dx}t(t(x)) &= 9a^4(a-1)^2\big(1-t(x)\big)^2(1-x)^2 \ge 0,\\
\frac{d^2}{dx^2}t(t(x)) &= 18a^4(a-1)^2(1-x)\big(1-t(x)\big)\big(4t(x)-1\big).
\end{aligned}$$

For $x \in (A,1]$ we obtain $4t(x)-1 < 4A-1 \le 0$, provided that $a \in [1, 4/3]$. Hence, $\frac{d^2}{dx^2}t(t(x)) < 0$ for all $x \in (A,1)$. Furthermore, $\frac{d}{dx}t(t(x))\big|_{x=A} = 9(a-1)^2 \le 1$ for $a \in \big[1, \frac{4}{3}\big]$. Thus, $\frac{d}{dx}t(t(x)) < 1$ for all $x \in (A,1)$. Therefore, the only fixed point of $[A,1] \ni x \mapsto t(t(x)) \in \mathbb{R}$ is A.

Let $(s_k)_{k=0}^{\infty}$ be given by $s_0 = 0$ and $s_k = t^k(s_0)$ for $k \in \mathbb{N}$. Clearly, $(s_{2k})_{k=0}^{\infty}$ is strictly increasing with $s_{2k} < A$ for $k \in \mathbb{N}_0$ and $(s_{2k+1})_{k=0}^{\infty}$ is strictly decreasing with $s_{2k+1} > A$ for $k \in \mathbb{N}_0$. We have

$$x_j \ge s_0 \quad \text{for} \quad j \in \{n_0, \dots, n_1 - 1\}.$$

Suppose

$$x_j \ge s_{2k} \quad \text{for} \quad j \in \{n_{2k}, \dots, n_{2k+1} - 1\}.$$

Then using (8) we obtain

$$\begin{aligned}
x_{n_{2k+1}+2} &= a x_{n_{2k+1}+1}(1 - x_{n_{2k+1}-1}) = a^2 x_{n_{2k+1}}(1 - x_{n_{2k+1}-1})(1 - x_{n_{2k+1}-2})\\
&= a^3 x_{n_{2k+1}-1}(1 - x_{n_{2k+1}-3})(1 - x_{n_{2k+1}-2})(1 - x_{n_{2k+1}-1}) \le a^3 A(1 - s_{2k})^3 = s_{2k+1}.
\end{aligned}$$

Similarly,

$$x_j \le s_{2k+1} \quad \text{for} \quad j \in \{n_{2k+1}, \dots, n_{2k+2} - 1\}$$

implies

$$\begin{aligned}
x_{n_{2k+2}+2} &= a x_{n_{2k+2}+1}(1 - x_{n_{2k+2}-1}) = a^2 x_{n_{2k+2}}(1 - x_{n_{2k+2}-2})(1 - x_{n_{2k+2}-1})\\
&= a^3 x_{n_{2k+2}-1}(1 - x_{n_{2k+2}-3})(1 - x_{n_{2k+2}-2})(1 - x_{n_{2k+2}-1}) \ge a^3 A(1 - s_{2k+1})^3 = s_{2k+2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
x_j &\in [s_{2k}, A] \quad \text{for} \quad j \in \{n_{2k}, \dots, n_{2k+1} - 1\},\\
x_j &\in [A, s_{2k+1}] \quad \text{for} \quad j \in \{n_{2k+1}, \dots, n_{2k+2} - 1\}
\end{aligned}$$

for all $k \in \mathbb{N}_0$. Clearly, $s_k \to A$ implies $x_k \to A$.

As $(s_{2k+1})_{k=0}^{\infty}$ is a decreasing sequence in $[A,1]$, $s_{2k+3} = t(t(s_{2k+1}))$, and A is the only fixed point of $t(t(x))$ in $[A,1]$, we obtain $s_{2k+1} \to A$. The continuity of t and $s_{2k+2} = t(s_{2k+1})$ gives $s_{2k} \to A$. □

We remark that the technique of [11, 7] seems to work for (1) to get global stability for $1 < a < 14/9$. However, [11, 7] does not apply directly, some additional work is necessary. Note that Proposition 5 is essential in the sense that the case $a \approx 1$ can not be handled in the computer-aided part of the method. The two fixed points can be arbitrarily close to each other, so after some point they can not be handled efficiently by interval arithmetic tools. Moreover, if they get closer to each other, they can not even be distinguished in the floating point system.

In the rest of the paper we assume $a \in (4/3, a_0]$.

3 An attracting neighborhood with linearization

For each fixed a, translating u_A into $0 \in \mathbb{R}^3$, i.e., introducing the new variable $v = u - u_A$, the shifted version

$$\mathbb{R}^3 \ni v \mapsto F_a(v + u_A) - u_A \in \mathbb{R}^3$$

of (2) can be written as

$$\mathbb{R}^3 \ni v \mapsto J_a v + f_a(v) \in \mathbb{R}^3, \tag{9}$$

where

$$J_a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-a & 0 & 1 \end{pmatrix}, \quad f_a(v) = \begin{pmatrix} 0 \\ 0 \\ -av_1v_3 \end{pmatrix}.$$

The characteristic polynomial of J_a is

$$P(a,\lambda) = -\lambda^3 + \lambda^2 + 1 - a,$$

and the roots of $P(a,\lambda)$ are $\lambda_i = \lambda_i(a)$, where $i \in \{0,1,2\}$. Since $P(1,\lambda) = -\lambda^2(\lambda-1)$ and $P(31/27,\lambda) = -(\lambda+1/3)(\lambda-2/3)^2$, it follows from the graph of $P(a,\lambda)$ that for $a \in (1, 31/27]$ the polynomial $P(a,\lambda)$ has three real roots (counting multiplicity) and $|\lambda_i| < 1$ for $i \in \{0,1,2\}$. For $a > 31/27$ the characteristic polynomial has a real root $\nu = \nu(a) = \lambda_0$ and two complex roots $\lambda_1 = \bar{\lambda}_2$. Denote λ the complex root with positive imaginary part, i.e., $\lambda = \lambda(a) = \lambda_1$. Formulas of ν and λ can be found in the Appendix, see Section 10.1.

From the graph of $P(a,\lambda)$ it follows that $\nu < 0$ and ν is a strictly decreasing function of a for $a > 31/27$. Since $P(3,\lambda) = -(\lambda+1)(\lambda-1-i)(\lambda-1+i)$, it is clear that $|\nu| < 1$ for $a \in (31/27, 3)$. From Vieta's formulas it follows that

$$\nu + 2\operatorname{Re}\lambda = 1, \qquad 2\nu\operatorname{Re}\lambda + |\lambda|^2 = 0, \qquad \nu|\lambda|^2 = 1 - a.$$

From the first two formulas we get

$$|\lambda|^2 = \nu^2 - \nu. \tag{10}$$

Combining the facts that $\mathbb{R} \ni s \mapsto s^2 - s \in \mathbb{R}$ is decreasing on $(-\infty, 0]$, and $\nu(a)$ is a decreasing function of a, we obtain $|\lambda(a)|$ is a strictly increasing function of a. From (10) and the third Vieta's formula, it follows that $|\lambda| = 1$ implies $\nu = \left(1-\sqrt{5}\right)/2$ and $a = a_0$. Thus, we obtain for $a \in (1, a_0)$ that $|\lambda_i| < 1$, $i \in \{0,1,2\}$, i.e., the fixed point u_A of F_a is locally stable. For $a > a_0$ we get $|\lambda| > 1$, so the fixed point is unstable.

The eigenvectors q_i, corresponding to the eigenvalues λ_i of J_a, are $q_i = q_i(a) = (1, \lambda_i, \lambda_i^2)$, $i \in \{0,1,2\}$. Let Q_a be the matrix, whose columns are the eigenvectors q_i, i.e.,

$$Q_a = \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \bar{\lambda} & \nu \\ \lambda^2 & \bar{\lambda}^2 & \nu^2 \end{pmatrix}.$$

For $a > 31/27$ the matrix Q_a is invertible and

$$Q_a^{-1} = \begin{pmatrix} d\,(\lambda^2 - \lambda) & d\,(\lambda - 1) & d \\ \bar{d}\,(\bar{\lambda}^2 - \bar{\lambda}) & \bar{d}\,(\bar{\lambda} - 1) & \bar{d} \\ e\,(\nu^2 - \nu) & e\,(\nu - 1) & e \end{pmatrix}, \tag{11}$$

where

$$e = e(a) = \frac{1}{3\nu^2 - 2\nu}, \quad d = d(a) = \frac{1}{3\lambda^2 - 2\lambda}.$$

Note that the rows of Q_a^{-1} are the eigenvectors of J_a^T.

Applying the linear transformation

$$\mathcal{Z} = \begin{pmatrix} z \\ \bar{z} \\ y \end{pmatrix} = Q_a^{-1} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

the map (9) takes the following form

$$\mathcal{Z} \mapsto Q_a^{-1} J_a Q_a \mathcal{Z} + Q_a^{-1} f_a\big(Q_a \mathcal{Z}\big) = \begin{pmatrix} \lambda\, z + d\, g(z,y) \\ \bar{\lambda}\, \bar{z} + \bar{d}\, g(z,y) \\ \nu\, y + e\, g(z,y) \end{pmatrix}, \tag{12}$$

where the function $g : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ is

$$g(z,y) = g_a(z,\bar{z},y) = -a(z+\bar{z}+y)\left(\lambda^2 z + \bar{\lambda}^2 \bar{z} + \nu^2 y\right). \tag{13}$$

Clearly, the second component in (12) is the complex conjugate of the first one. Therefore, it is sufficient to consider the map

$$H_a :\ \mathbb{C} \times \mathbb{R} \ni \begin{pmatrix} z \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda z + d\, g(z,y) \\ \nu y + e\, g(z,y) \end{pmatrix} \in \mathbb{C} \times \mathbb{R}. \tag{14}$$

Remark that for the sake of simplicity, we omit the argument $\bar{z}$, and indicate only variable z in the subsequent functions. We also emphasize that $g(z,y)$ and $H_a(z,y)$ are smooth functions of $z, \bar{z}$ and y, but they are not necessarily complex differentiable.

3.1 Local stability by linearization

First, for a fixed parameter $a \in (4/3, a_0)$ we use the map (14) without further transformation to construct a neighborhood $\mathcal{M}_0(a) \subseteq \mathbb{C} \times \mathbb{R}$, which is inside the region of attraction of the origin, i.e., $\lim_{n\to\infty} H_a^n(z,y) = (0,0)$ for every $(z,y) \in \mathcal{M}_0(a)$.

Proposition 6. *For every $a \in (4/3, a_0)$ define $\xi(a)$ by*

$$\xi(a) = \frac{(1-|\lambda|)}{4a|\lambda|^2(|d|+|e|)}.$$

Then the set

$$\mathcal{M}_0(a) = \{(z,y) \in \mathbb{C} \times \mathbb{R} :\ |z|+|y| < \xi(a)\}$$

is in the region of attraction of the fixed point $(0,0)$ of H_a.

Proof. Introduce the norm $|(z,y)| = |z|+|y|$ on the space $\mathbb{C} \times \mathbb{R}$. We show that there exists a $\xi = \xi(a) > 0$ such that $|H_a(z,y)| < |(z,y)|$ for every $0 < |(z,y)| < \xi$. If such a ξ exists, it is clear that the open ball B_ξ° around the origin is invariant. We show that every point of B_ξ° tends to the origin. Let (z_0,y_0) be an arbitrary point from B_ξ° and consider the nonnegative, strictly decreasing sequence $(|(z_n,y_n)|)_{n=0}^\infty$, where $(z_{n+1},y_{n+1}) = H_a(z_n,y_n)$. This sequence can converge only to a fixed point of the continuous map $r \mapsto \max_{|(z,y)|=r} |H_a(z,y)|$, which is only $r=0$, provided that $r \in [0,\xi)$.

From (10) we obtain $|\lambda| > |\nu|$ for $4/3 < a < a_0$, so

$$|g(z,y)| \le 4a|\lambda|^2(|z|+|y|)^2 = 4a|\lambda|^2|(z,y)|^2.$$

Consequently, estimating (14) we obtain

$$\begin{aligned} |H_a(z,y)| &\le |\lambda||(z,y)| + (|d|+|e|)\,|g(z,y)| \\ &\le |(z,y)|\,\big(|\lambda| + (|d|+|e|)\,4a|\lambda|^2|(z,y)|\big) < |(z,y)| \end{aligned}$$

for every $(z, y) \neq (0, 0)$, provided that $|(z, y)| < \xi(a)$, where

$$\xi(a) = \frac{1 - |\lambda|}{4a|\lambda|^2(|d| + |e|)}.$$

Therefore, $\xi(a)$ is a suitable choice. $\square$

First, note that $\mathcal{M}_0(a)$ is in $\mathbb{C} \times \mathbb{R}$. Clearly, this set corresponds to an attracting neighborhood $\mathcal{M}(a) \subseteq \mathbb{R}^3$ around the nontrivial fixed point u_A of (2). However, we let this transformation be done in the second, computer-aided part of the proof, in order to obtain better accuracy.

Second, since $\lim_{a \to a_0} |\lambda(a)| = 1$, we obtain $\mathcal{M}_0(a)$ shrinks to the origin as a tends to a_0. However, the smaller the neighborhood is, the less efficient and more time-consuming the computer-aided part of the proof is. Furthermore, Proposition 6 does not provide at all an attractive neighborhood at the critical parameter value a_0. Consequently, close to a_0 this approach is not suitable for reliable numerical methods, and thus we need to find another way to construct the attracting neighborhood.

4 A center manifold reduction

In the subsequent sections we study the case $a \in \mathcal{I}_0 = [a_0 - 10^{-2}, a_0]$. For $a \in \mathcal{I}_0$ we want to adapt the normal form technique from [3] to create an attracting neighborhood around the nontrivial fixed point of map (2). However, map (2) is 3-dimensional, so we need a center manifold reduction first (see [12, 13, 14]).

As we explained in the Introduction, a polynomial approximation of the generalized center-unstable manifold will be used here for each $a \in \mathcal{I}_0$. We look for the fourth order polynomial approximation $\phi(z) = \phi_a(z, \bar{z})$ of the manifold in the form

$$\phi(z) = \sum_{n=2}^{4} \sum_{i+j=n} \frac{1}{i!j!} \omega_{ij} z^i \bar{z}^j, \tag{15}$$

where $\omega_{ij} = \omega_{ij}(a)$. Every coefficient ω_{ij} is determined so that in the expression

$$\mathcal{N}(\phi(z)) = \phi\Big(\lambda z + d\, g\big(z, \phi(z)\big)\Big) - \nu\phi(z) - e\, g\big(z, \phi(z)\big) \tag{16}$$

the at most fourth order terms of $z, \bar{z}$ are eliminated (see [13]), so $\mathcal{N}(\phi(z)) = O\left(|z|^5\right)$. The coefficients ω_{ij} depend smoothly on a and the formulas can be found in the Appendix, see Section 10.3. In the subsequent sections we study the dynamics of (14) in the set

$$T(r, C) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \leq r, \ |y - \phi(z)| \leq C|z|^5\}.$$

Note that it would suffice to consider a second order approximation of the generalized center-unstable manifold with a term $C|z|^3$ in $T(r, C)$ in order to obtain an attracting neighborhood. Also in this case C would appear only in the at least fourth order terms of (4) which is of crucial importance. However, this does not provide a sufficiently large neighborhood for the computer-aided part of the method. Indeed, the larger the order of the approximation of the center manifold is, the larger the obtainable initial attracting neighborhood is, since our estimates are more precise. On the other hand, calculations become lengthier as the order of the approximation increases. Furthermore, the growth of attracting neighborhood due to extra orders is diminishing, that is why, we have chosen a fourth order approximation for $\phi(z)$.

Throughout the article we need an a-independent estimate of some coefficients. First of all, it is easy to see that

$$|\lambda| \le 1 \quad \text{and} \quad |\nu| \le \nu_0 \qquad \text{for } a \in \mathcal{I}_0, \tag{17}$$

where $\nu_0 = a_0 - 1$. Furthermore, with interval arithmetic it can be shown that $d_0 = 0.559$ and $e_0 = 0.425$ satisfy the inequalities

$$|d| \le d_0 \quad \text{and} \quad |e| \le e_0 \qquad \text{for } a \in \mathcal{I}_0. \tag{18}$$

Then, we choose constants ω_k, $k \in \{2,3,4\}$ such that the following holds

$$\sum_{i+j=k} \frac{|\omega_{ij}|}{i!j!} \le \omega_k \qquad \text{for } a \in \mathcal{I}_0.$$

With interval arithmetic we obtain that $\omega_2 = 1.29$, $\omega_3 = 2.193$ and $\omega_4 = 6.233$ are appropriate choices. Let $\phi_k(z)$ denote the k-th order terms of $\phi(z)$. We obtain

$$|\phi_k(z)| \le \omega_k r^k \tag{19}$$

for $k \in \{2,3,4\}$ and $a \in \mathcal{I}_0$. Clearly, (19) implies the polynomial estimate

$$|\phi(z)| \le \phi^{max}(|z|) \tag{20}$$

for every $a \in \mathcal{I}_0$, where

$$\phi^{max}(r) = \sum_{k=2}^{4} \omega_k r^k$$

is a real polynomial with positive coefficients. However, estimate (19) is a stronger property than merely (20). We refer to an estimate like (20) as an *estimate by order*, if inequalities similar to (19) also holds.

Now, we turn our attention to the function $g(z,y)$ from (13) and consider the expansion

$$g(z,y) = \sum_{i+j+k=2} \frac{g_{ijk}}{i!j!k!} z^i \bar{z}^j y^k, \tag{21}$$

where

$$g_{ijk} = g_{ijk}(a) = -a\left(i\lambda^2 + j\bar{\lambda}^2 + k\nu^2\right).$$

Using interval arithmetics it can be shown that $g_{20} = 4.237$, $g_{11} = 3.805$ and $g_{02} = 0.6181$ satisfy

$$\sum_{i+j=2} \frac{|g_{ij0}|}{i!j!} \le g_{20}, \qquad \sum_{i+j=1} \frac{|g_{ij1}|}{i!j!} \le g_{11}, \qquad \frac{|g_{002}|}{2!} \le g_{02} \quad \text{for } a \in \mathcal{I}_0.$$

Thus, from (21) we obtain

$$|g(z,y)| \le \tilde{g}^{max}(|z|,|y|) \tag{22}$$

for every $a \in \mathcal{I}_0$, where

$$\tilde{g}^{max}(r,s) = g_{20}r^2 + g_{11}rs + g_{02}s^2.$$

Here, we also estimate by order (distinguishing y from z and $\bar{z}$, but handling z and $\bar{z}$ together), and the coefficients of the second order polynomial $g^{max}(r,s)$ are positive.

Later, we need also the composition of functions ϕ and g, so we introduce the 8th order polynomial

$$g^{max}(r) = \tilde{g}^{max}\big(r, \phi^{max}(r)\big),$$

and for every $a \in \mathcal{I}_0$ we obtain

$$|g(z, \phi(z))| \leq g^{max}(|z|). \tag{23}$$

It is important that (23) is also an estimate by order, since it is a composition of two functions with that property.

After these estimations we can proceed to show the conditional invariance of $T(r, C)$. First of all, $y = \phi(z)$ is just an approximation of the center manifold, so $\mathcal{N}(\phi(z))$ is not zero. Nevertheless, $\mathcal{N}(\phi(z)) = O(|z|^5)$ follows from the choice of coefficients ω_{ij}. In the computer-aided part, for a given $\rho_1 > 0$ we need an explicit $\mathcal{N}(\phi(z)) \leq C_0|z|^5$-type estimate for $|z| \leq \rho_1$ and $a \in \mathcal{I}_0$, where the constant C_0 is independent of a and z.

Proposition 7. *Let $\rho_1 = 0.02234$ and $C_0 = 37.379$. For every $|z| \leq \rho_1$ and $a \in \mathcal{I}_0$ the following holds*

$$|\mathcal{N}(\phi(z))| \leq C_0|z|^5.$$

Proof. Because of the construction of $\phi(z)$, the fourth and lower order terms of $z, \bar{z}$ are zero in (16). To gain a better estimate of C_0, we consider the decomposition

$$\mathcal{N}(\phi(z)) = \mathcal{N}_5(z) + \mathcal{N}_{\geq 6}(z),$$

where $\mathcal{N}_5(z)$ and $\mathcal{N}_{\geq 6}(z)$ denote the fifth and the at least sixth order terms of $\mathcal{N}(\phi(z))$, respectively. Clearly, $\mathcal{N}_5(z)$ can be written in the following form

$$\mathcal{N}_5(z) = \sum_{i+j=5} \frac{N_{ij}}{i!j!} z^i \bar{z}^j,$$

where $N_{ij} = N_{ij}(a)$. Using (15) and (21) the coefficients N_{ij} can be determined explicitly for $i + j = 5$. The formulas can be found in the Appendix, see Section 10.4. With interval arithmetic it can be shown that $N_5 = 25.094$ satisfies

$$\sum_{i+j=5} \frac{|N_{ij}|}{i!j!} \leq N_5 \tag{24}$$

for every $a \in \mathcal{I}_0$. From this we obtain $|\mathcal{N}_5(z)| \leq N_5|z|^5$ for $a \in \mathcal{I}_0$.

For the estimation of $\mathcal{N}_{\geq 6}(z)$ we use (17), (18), (20) and (23). For every $a \in \mathcal{I}_0$ we get

$$|\mathcal{N}(\phi(z))| \leq \phi^{max}\big(r + d_0\, g^{max}(r)\big) + \nu_0\, \phi^{max}(r) + e_0\, g^{max}(r), \tag{25}$$

where $r = |z|$. The right hand side of (25) is a polynomial, so it can be written in the form

$$\phi^{max}\big(r + d_0\, g^{max}(r)\big) + \nu_0\, \phi^{max}(r) + e_0\, g^{max}(r) = \sum_{k=2}^{32} b_{0,k} r^k.$$

The a-independent real coefficients $b_{0,k}$ can be determined by the real polynomials $\phi^{max}(r)$ and $g^{max}(r)$. The inequality (25) is also an estimate by order, since (20) and (23) also have that property. Furthermore, the lower order terms of $\mathcal{N}(\phi(z))$ were eliminated, so the following also holds

$$|\mathcal{N}_{\geq 6}(z)| \leq \sum_{k=6}^{32} b_{0,k} r^k.$$

Note that this inequality would not be necessarily true without the estimate of order property.

Using $r \leq \rho_1$ we obtain

$$|\mathcal{N}_{\geq 6}(z)| \leq \left(\sum_{k=6}^{32} b_{0,k} \rho_1^{k-5} \right) |z|^5.$$

It can be show that $N_6 = 12.285$ satisfies

$$\sum_{k=6}^{32} b_{0,k} \rho_1^{k-5} \le N_6. \tag{26}$$

Combining (24) and (26) we obtain

$$|\mathcal{N}(\phi(z))| \le |\mathcal{N}_5(z)| + |\mathcal{N}_{\ge 6}(z)| \le (N_5 + N_6)|z|^5.$$

So the proposition is proven, since C_0 was chosen such that $N_5 + N_6 \le C_0$. □

In the following corollary we reformulate Proposition 7 in order to obtain a geometrical interpretation of the statement (see Figure 4).

Corollary 8. *Let ρ_1 and C_0 be from Proposition 7. For every $|z_0| \le \rho_1$ and $a \in \mathcal{I}_0$ the point $(\hat{z}_0, \hat{y}_0) = H_a(z_0, \phi(z_0))$ satisfies the inequality*

$$|\phi(\hat{z}_0) - \hat{y}_0| \le C_0 |z_0|^5.$$

4.1 Attractivity in direction y

Corollary 8 states that for parameter values close to the critical a_0 if we consider a point (z_0, y_0) from the surface $(z, \phi(z))$, i.e., $y_0 = \phi(z_0)$, then the image $(\hat{z}_0, \hat{y}_0)$ remains close to that surface. Now, with the help of Corollary 8 we are able to make a statement about the y-directional behavior of map (14). It is well-known (see [14]) that if the fixed point has no eigenvalues moduli greater than one, then the center manifold has an attracting property, i.e., every solution close enough to the fixed point decays exponentially to the center manifold. Based on this idea we prove a similar statement about the approximation $y = \phi(z)$ of the center manifold.

Proposition 9. *Let ρ_1 and C_0 be from Proposition 7. Furthermore, let $\tilde{\rho}_2 = 0.0237$, $\sigma = 2.1 \cdot 10^{-3}$ and $L = 0.66$. For every $|z_0| \le \rho_1$, $|y_0| \le \sigma$ and $a \in \mathcal{I}_0$ the point $(z_1, y_1) = H_a(z_0, y_0)$ satisfies $|z_1| \le \tilde{\rho}_2$ and*

$$|\phi(z_1) - y_1| \le L|\phi(z_0) - y_0| + C_0|z_0|^5.$$

Proof. First, note that σ was chosen large enough such that Proposition 9 can be applied in Propositions 10, 12 and 13, i.e., $\{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \le \rho_1, \ |y| \le \sigma\}$ contains the occurring T and $\tilde{T}$. Second, from (14), (17), (18) and (22) it is clear that

$$|z_1| \le |z_0| + d_0 \, \tilde{g}^{max}(|z_0|, |y_0|)$$

for every $a \in \mathcal{I}_0$. The constant $\tilde{\rho}_2$ was chosen such that

$$\tilde{\rho}_2 \ge \rho_1 + d_0 \, \tilde{g}^{max}(\rho_1, \sigma).$$

Thus, we obtain that $|z_1| \le \tilde{\rho}_2$ for every $|z_0| \le \rho_1$ and $|y_0| \le \sigma$. Similarly, $|\hat{z}_0| \le \tilde{\rho}_2$ also holds for $(\hat{z}_0, \hat{y}_0) = H_a(z_0, \phi(z_0))$, since $\phi^{max}(\rho_1) \le \sigma$ (see Corollary 8 and Figure 4). Finally, introducing the notation $k_i = y_i - \phi(z_i)$ for $i \in \{0, 1\}$ (see Figure 4), the formula to be proven in this proposition can be reformulated into the form

$$|k_1| \le L|k_0| + C_0|z_0|^5.$$

Using (14) and the mean value theorem we obtain

$$\begin{aligned} z_1 &= \lambda z_0 + d\, g\big(z_0, \phi(z_0) + k_0\big) = \hat{z}_0 + d\, k_0\, \partial_2 g\big(z_0, \phi(z_0) + \tilde{k}_0\big), \\ y_1 &= \nu \left(\phi(z_0) + k_0\right) + e\, g\big(z_0, \phi(z_0) + k_0\big) = \hat{y}_0 + \nu\, k_0 + e\, k_0\, \partial_2 g\big(z_0, \phi(z_0) + \hat{k}_0\big), \end{aligned} \tag{27}$$

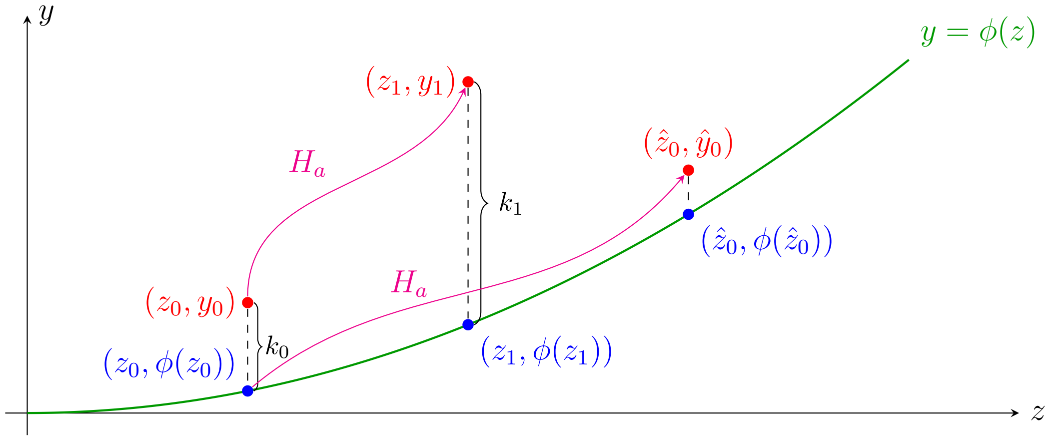


Figure 4. The dynamics close to $y = \phi(z)$

where

$$\partial_2 g(z,y) = -a\left((\lambda^2+\nu^2)z + (\bar{\lambda}^2+\nu^2)\bar{z} + 2\nu^2 y\right)$$

is the partial derivative of $g(z,y)$ with respect to y, and $\tilde{k}_0, \hat{k}_0$ are some numbers between 0 and k_0. Since $|\phi(z_0)+k_0| \le \sigma$ and $|\phi(z_0)| \le \sigma$, we get

$$|\phi(z_0)+\tilde{k}_0| \le \sigma, \qquad |\phi(z_0)+\hat{k}_0| \le \sigma. \tag{28}$$

From (22), we obtain

$$|\partial_2 g(z,y)| \le g_{11}|z| + 2g_{02}|y|$$

for every $a \in \mathcal{I}_0$. Introduce $\partial_y g$ such that $|\partial_2 g(z,y)| \le \partial_y g$ holds for every $|z| \le \rho_1$ and $|y| \le \sigma$. It can be shown that $\partial_y g = 0.088$ is a suitable choice. Using $\partial_y g$, (17), (18) and (28) we obtain from (27) that

$$|z_1 - \hat{z}_0| \le d_0\, \partial_y g\, |k_0|. \tag{29}$$

Similarly, from (27) we gain

$$|y_1 - \hat{y}_0| \le (\nu_0 + e_0\, \partial_y g)\, |k_0| \tag{30}$$

for every $|z_0| \le \rho_1$, $|y_0| \le \sigma$ and $a \in \mathcal{I}_0$.

It is easy to see that

$$|\phi(z_1) - \phi(\hat{z}_0)| \le |z_1 - \hat{z}_0| \partial\phi^{max}(r), \tag{31}$$

provided that $|z_1| \le r$ and $|\hat{z}_0| \le r$, where

$$\partial\phi^{max}(r) = 2\omega_2 r + 3\omega_3 r^2 + 4\omega_4 r^3.$$

Since $|z_0| \le \rho_1$ and $|y_0| \le \sigma$ implies $|z_1| \le \tilde{\rho}_2$ and $|\hat{z}_0| \le \tilde{\rho}_2$, we introduce $\partial\phi$ such that

$$\partial\phi^{max}(\tilde{\rho}_2) \le \partial\phi.$$

It can be shown that $\partial\phi = 0.066$ is a proper choice.

Using Corollary 8, (29), (30), (31) and the definition of $\partial\phi$ we gain

$$\begin{aligned}|k_1| = |y_1 - \phi(z_1)| &\le |y_1 - y_2| + |y_2 - \phi(z_2)| + |\phi(z_2) - \phi(z_1)| \\ &\le (\nu_0 + e_0\, \partial_y g)|k_0| + C_0|z_0|^5 + \partial\phi\, d_0\, \partial_y g|k_0| \\ &= (\nu_0 + e_0\, \partial_y g + \partial\phi\, d_0\, \partial_y g)|k_0| + C_0|z_0|^5\end{aligned}$$

for every $|z_0| \le \rho_1$, $|y_0| \le \sigma$ and $a \in \mathcal{I}_0$. It can be checked that

$$\nu_0 + e_0\, \partial_y g + \partial\phi\, d_0\, \partial_y g \le L,$$

so the proposition is proven. □

It can be seen that Proposition 9 is slightly weaker than similar statements about center manifolds. Even if a solution remains close to the fixed point, $y = \phi(z)$ does not have a real exponentially attractive property, since we have the extra term $C_0|z_0|^5$ in the estimate. This term originated from the fact that $y = \phi(z)$ is not the center manifold, but only an approximation of it. On the other hand, it is of crucial importance that we explicitly give the neighborhood where the proposition holds, and determine also an explicit value for the parameter L.

4.2 Conditional invariance in direction y

Using Proposition 9 we can show the y-directional conditional invariance of $T(\rho_1, C_1)$ for an appropriately chosen $C_1 > C_0$.

Proposition 10. *Let ρ_1 be from Proposition 7 and $C_1 = 7700$. For every $(z_0, y_0) \in T(\rho_1, C_1)$ and $a \in \mathcal{I}_0$ the point $(z_1, y_1) = H_a(z_0, y_0)$ satisfies*

$$|\phi(z_1) - y_1| \le C_1 |z_1|^5.$$

Proof. Using the notations from Proposition 9 it is clear that $|k_0| = |\phi(z_0) - y_0| \le C_1|z_0|^5$, since $(z_0, y_0) \in T(\rho_1, C_1)$. We need to prove the inequality $|k_1| \le C_1|z_1|^5$.

Since σ was chosen such that $\phi^{max}(\rho_1) + \rho_1^5 \le \sigma$, we can apply Proposition 9. We obtain $|z_1| \le \tilde{\rho}_2$ and

$$|k_1| \le L|k_0| + C_0|z_0|^5 \le (LC_1 + C_0)|z_0|^5.$$

However, we need a $|k_1| \le C|z_1|^5$-type inequality, so first, we are looking for a constant C_2 such that $|z_0|^5 \le C_2|z_1|^5$. From (27) we gain that

$$|z_1| \ge \lambda_{min}|z_0| - d_0\, \tilde{g}^{max}(|z_0|, |y_0|) \ge \lambda_{min}|z_0| - d_0\, \tilde{g}^{max}\big(|z_0|, \phi^{max}(|z_0|) + C_1|z_0|^5\big)$$

for every $(z_0, y_0) \in T(\rho_1, C_1)$, and furthermore, $\lambda_{min} = 0.9952$ was chosen so that it satisfies $\lambda_{min} \le |\lambda|$ for every $a \in \mathcal{I}_0$. Since $g^{max}\big(r, \phi^{max}(r) + C_1 r^5\big) = O(r^2)$, introducing

$$g_0^{max}(r) = \frac{\tilde{g}^{max}\big(r, \phi^{max}(r) + C_1 r^5\big)}{r},$$

where $g_0^{max}(r) = O(r)$, we obtain

$$|z_1| \ge |z_0|\big(\lambda_{min} - d_0\, g_0^{max}(|z_0|)\big) \ge |z_0|\big(\lambda_{min} - d_0\, g_0^{max}(\rho_1)\big).$$

Since $C_2 = 1.358$ satisfies

$$\frac{1}{\big(\lambda_{min} - d_0\, g_0^{max}(\rho_1)\big)^5} \le C_2,$$

we gain $|z_0|^5 \le C_2|z_1|^5$, and

$$|\phi(z_1) - y_1| = |k_1| \le (LC_1 + C_0)C_2|z_1|^5.$$

Using the values of C_0, C_1, C_2 and L it can be checked that

$$(LC_1 + C_0)C_2 \le C_1,$$

so the proposition is proven. □

It follows from Proposition 10 that if $|z_1| \le \rho_1$, then $(z_1, y_1) \in T(\rho_1, C_1)$. However, at this point we can guarantee only that $|z_1| \le \tilde{\rho}_2$, and thus

$$H_a(T(\rho_1, C_1)) \subseteq T(\tilde{\rho}_2, C_1).$$

Roughly speaking, for $T(\rho_1, C_1)$ we proved some kind of conditional invariance in direction y under the map (14), but we still do not have overall picture about the z-directional dynamics. It will be covered in the subsequent section using the bifurcational normal form technique adapted from [3].

Note that it also follows from Proposition 10 that $T(r, C_1)$ is also conditionally invariant in direction y for every $r \le \rho_1$. Finally, we remark that C_1 is not the smallest value for which Proposition 10 holds; it is about $C \approx 475$. However, we need a relatively thick (in direction y) set for the computer-aided part, and it is easier to construct a real neighborhood around the origin from a $T(r, C)$ with a larger C.

5 Transforming to normal form

In the previous section we saw that $T(\rho_1, C_1)$ is conditionally invariant in direction y under the map (14). Now, using the Neimark–Sacker bifurcational normal form technique from [3] we study the z-directional dynamics. Essentially, we perform a nonlinear transformation on the z-coordinate of the z–y-coordinate system such that (14) becomes a contraction (close to the fixed point) along the transformed direction in the new coordinate system. In the end, combining this with the y-directional dynamics we obtain that some subset of $T(\rho_1, C_1)$ is in the region of attraction of the fixed point of (14).

For every $(z_0, y_0) \in T(\rho_1, C_1)$ the y-coordinate can be written in the form $y_0 = \phi(z_0) + c|z_0|^5$ for some $c \in \mathbb{R}$ with $|c| \le C_1$. Thus, for $(z_1, y_1) = H_a(z_0, y_0)$ the z-coordinate is determined by $z_1 = G(z_0)$, where

$$G(z) = G_{a,c}(z, \bar{z}) = \lambda z + d\, g\big(z, \phi(z) + c|z|^5\big). \tag{32}$$

For every fixed $a \in \mathcal{I}_0$ and c with $|c| \le C_1$ we adapt the normal form technique to (32). However, we have an extra parameter c in (32) and an additional stable direction in (14) compared to [3]. Thus first, before the main result of this section we investigate in detail how this c effects the method from [3]. We also discuss the specific shape of $T(r, C)$, which assures that the aforementioned normal form technique can be adapted with minor changes. After that, the proof of Proposition 11 is essentially the same as in [3]. However, we elaborate it for the sake of completeness.

Introducing the sets

$$\tau(c) = \big\{\big(z, \phi(z) + c|z|^5\big) : z \in \mathbb{C}\big\}$$
$$\tau(r, c) = \big\{\big(z, \phi(z) + c|z|^5\big) : z \in \mathbb{C},\ |z| \le r\big\}$$

we can note that for a fixed $a \in \mathcal{I}_0$ the set $T(\rho_1, C_1)$ is foliated by the sets $\tau(\rho_1, c)$ with $|c| \le C_1$. That is, $T(\rho_1, C_1) = \cup_{|c| \le C_1} \tau(\rho_1, c)$ and $\tau(\rho_1, c) \cap \tau(\rho_1, \hat{c}) = \{(0, 0)\}$ for every $c \ne \hat{c}$ with $|c| \le C_1$ and $|\hat{c}| \le C_1$. Using (32) and the normal form technique we consider the dynamics on $\tau(\rho_1, c)$ for a fixed $a \in \mathcal{I}_0$ and $|c| \le C_1$. However, $\tau(\rho_1, c)$ is not invariant under (14), more precisely $H_a(\tau(\rho_1, c)) \nsubseteq \tau(c)$ in general. Consequently, proving the contraction of (32) for every a and c separately, could cause a lot of difficulty if there is no connection between the methods for different parameter values. The reason is that the nonlinear transformation of the z-coordinate and the neighborhood on which the transformed map is a contraction could vary from value to value. Therefore, considering $a \in \mathcal{I}_0$ fixed, our aim is to handle these transformations together in some sense for every c with $|c| \le C_1$.

Following the steps from [3] the function $G(z)$ in (32) can be written as a formal Taylor series of complex variables z and $\bar{z}$, i.e.,

$$G(z) = \lambda z + \sum_{2 \le i+j \le 5} \frac{G_{ij}}{i!j!} z^k \bar{z}^l + R_1, \tag{33}$$

where $G_{ij} = G_{ij}(a)$ and $R_1 = R_1(z, \bar{z}, a, c) = O(|z|^6)$. The expression of G_{ij} can be found in the Appendix, see Section 10.5. Observe that, because of the definition of the set $T(r, C)$, the parameter c appears only in the at least sixth order terms of G, i.e., G_{ij} is independent of c for $2 \le i+j \le 5$.

It is well known (see e.g., [3, 12]) that there is a locally invertible parameter-dependent change of the complex coordinate $z = h(w)$ with $h : \mathbb{C} \to \mathbb{C}$ in the form

$$h(w) = w + \frac{h_{20}}{2} w^2 + h_{11} w \overline{w} + \frac{h_{02}}{2} \overline{w}^2 + \frac{h_{30}}{6} w^3 + \frac{h_{12}}{2} w \overline{w}^2 + \frac{h_{03}}{6} \overline{w}^3, \tag{34}$$

so that the coefficients $h_{ij} = h_{ij}(a)$ are independent of c, and the map (32) is transformed into its normal form

$$w \mapsto h^{-1}(G(h(w))) = \lambda w + c_1 w^2 \overline{w} + R_2, \tag{35}$$

where $c_1 = c_1(a)$ is the Lyapunov-coefficient and $R_2 = R_2(w, \overline{w}, a, c) = O(|w|^4)$. The inverse h^{-1} of h can be defined in a small neighborhood of $0 \in \mathbb{C}$ in the form $h_0^{-1}(z) + O(|z|^6)$, where

$$h_0^{-1}(z) = z + \sum_{2 \le i+j \le 5} \frac{\tilde{h}_{ij}}{i!j!} z^i \bar{z}^j \tag{36}$$

with c-independent coefficients $\tilde{h}_{ij} = \tilde{h}_{ij}(a)$. The existence of h with the properties above requires the nonresonance condition

$$\left(\frac{\lambda(a)}{|\lambda(a)|} \right)^k \ne 1$$

for $k \in \{1, 2, 3, 4\}$ and $a \in \mathcal{I}_0$, which obviously holds. The coefficients $\tilde{h}_{ij}$ of h_0^{-1} can be obtained from the equation $z = h\left(h_0^{-1}(z)\right) + O(|z|^6)$ by equating the coefficients of the same type up to fifth order, see Section 10.7. The coefficient h_{ij} of h are chosen so that the second and third order terms of $h_0^{-1}(G(h(w)))$ are eliminated (apart from $w^2\overline{w}$), see Section 10.6. As the coefficients G_{ij} with $2 \le i+j \le 5$ are independent of c, it is not difficult to see that h_{ij} and $\tilde{h}_{ij}$ are also independent of c.

The inverse h^{-1} will be given in Subsection 5.2 as the inverse of the restriction of h to a neighborhood of $0 \in \mathbb{C}$. The main issue here is the construction of

$$h^{-1}(z) = h_0^{-1} + R_3 \tag{37}$$

with some $R_3 = R_3(z, \bar{z}, a) = O(|z|^6)$, and an explicit bound for R_3 on its domain uniformly in $a \in \mathcal{I}_0$ and $c \in \mathbb{R}$ with $|c| \le C_1$.

A key fact is that for each fixed $a \in \mathcal{I}_0$ the transformations h, h^{-1} are the same for all c with $|c| \le C_1$. Therefore, the transformation $z = h(w)$ is the same for the whole set $T(\rho_1, C_1)$. The c-dependence appears only in R_2 in the transformed map (35).

Since a supercritical bifurcation takes place at a_0, it can be shown that for every c with $|c| \le C_1$ and $a \le a_0$ sufficiently close to a_0 there exists some $\rho_0 = \rho_0(a, c) > 0$ such that for every $w \in \mathbb{C}$ with $0 < |w| \le \rho(a, c)$ the inequality

$$\left| \lambda w + c_1 w^2 \overline{w} + R_2 \right| < |w| \tag{38}$$

holds. Our aim is to obtain a uniform ρ_0 for all $a \in \mathcal{I}_0$ and $|c| \leq C_1$ such that this ρ_0 is sufficiently large for the rigorous computational part of the method. Note that inequality (38) shed some light on why we did not choose initially a constant neighborhood around $y = \phi(z)$.

The elaboration and combination of the ideas above lead to the following result.

Proposition 11. *Let C_1 be from Proposition 10 and $\varepsilon_G = 0.01976$. For every $a \in \mathcal{I}_0$ the set $T(\varepsilon_G, C_1)$ is in the region of attraction of the fixed point of map* (14).

Proof. The rest of this section is devoted to the proof of Proposition 11 following the steps of [3]. First, let $a \in \mathcal{I}_0$ and c with $|c| \leq C_1$ be fixed.

5.1 Estimation of the lower order terms in G, h and h_0^{-1}

Throughout this section we need an estimate of the coefficients of the lower order terms in G, h and h_0^{-1} such that these estimates are independent of $a \in \mathcal{I}_0$. Using (33), (34) and (36) we look for constants satisfying the inequalities

$$\begin{aligned} \sum_{i+j=n} \frac{|G_{ij}|}{i!j!} &\leq G_n, \\ \sum_{i+j=2} \frac{|h_{ij}|}{i!j!} &\leq h_2, \\ \frac{|h_{30}|}{6} + \frac{|h_{12}|}{2} + \frac{|h_{03}|}{6} &\leq h_3, \\ \sum_{i+j=n} \frac{|\tilde{h}_{ij}|}{i!j!} &\leq \tilde{h}_n \end{aligned} \tag{39}$$

for every $a \in \mathcal{I}_0$ and $n \in \{2,3,4,5\}$. With interval arithmetic it can be shown that $G_2 = 2.341$, $G_3 = 2.352$, $G_4 = 3.955$, $G_5 = 11.237$, $h_2 = \tilde{h}_2 = 2.93$, $h_3 = 4.976$, $\tilde{h}_3 = 10.353$, $\tilde{h}_4 = 34.796$ and $\tilde{h}_5 = 110.572$ satisfy the requirements. From the definition of these constants we obtain the following finite-order polynomial estimates of functions h and h_0^{-1}

$$|h(w)| \leq h^{max}(|w|), \qquad \left|h_0^{-1}(z)\right| \leq \tilde{h}_0^{max}(|z|), \tag{40}$$

where

$$\begin{aligned} h^{max}(r) &= r + h_2 r^2 + h_3 r^3, \\ \tilde{h}_0^{max}(r) &= r + \tilde{h}_2 r^2 + \tilde{h}_3 r^3 + \tilde{h}_4 r^4 + \tilde{h}_5 r^5. \end{aligned} \tag{41}$$

We emphasize that in (40) we estimate by order and in (41) the coefficients are all independent of a and c.

From the definition of h_2 and h_3 we also get

$$|w| - h_2|w|^2 - h_3|w|^3 \leq |h(w)|. \tag{42}$$

Consequently, assuming $|w| \leq \rho$ and $h_2\rho + h_3\rho^3 < 1$ we obtain

$$|w| \leq \eta(\rho)|h(w)| \tag{43}$$

with

$$\eta(\rho) = \frac{1}{1 - h_2\rho - h_3\rho^2}.$$

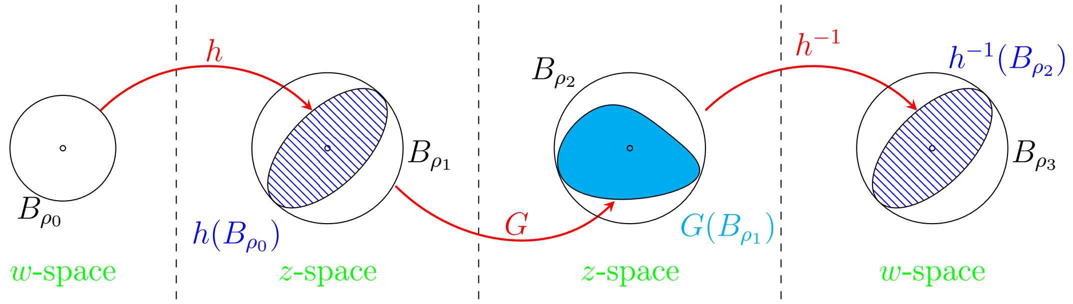


Figure 5. The size of the domains of h, G and h^{-1}, provided that $|w| \le \rho_0$

5.2 The definition of h^{-1}

Set $\delta_1 = 1/10$ and $\delta_2 = 1/16$. Our aim is to show that h^{-1} can be defined on $\overline{B}_{\delta_2} \subseteq \mathbb{C}$. For a fixed $a \in \mathcal{I}_0$ and $z \in \mathbb{C}$ define $H_{a,z} : \mathbb{C} \ni w \mapsto w + z - h(w) \in \mathbb{C}$. Then there is a $w \in \mathbb{C}$ with $h(w) = z$ if and only if $H_{a,z}(w) = w$. From (40) and (41), for all $z, w_1, w_2 \in \mathbb{C}$ we obtain

$$\begin{aligned}|H_{a,z}(w_1) - H_{a,z}(w_2)| &= |w_1 - h(w_1) - w_2 + h(w_2)| \\ &\le |w_1 - w_2| \Big(h_2(|w_1| + |w_2|) + h_3 \left(|w_1|^2 + |w_1| \cdot |w_2| + |w_2|^2\right) \Big).\end{aligned}$$

If $z \in \overline{B}_{\delta_2}$ and $w_1, w_2 \in \overline{B}_{\delta_1}$, then

$$|H_{a,z}(w_1) - H_{a,z}(w_2)| \le |w_1 - w_2| \left(2\delta_1 h_2 + 3\delta_1^2 h_3\right).$$

In addition, for $z \in \overline{B}_{\delta_2}$ and $w \in \overline{B}_{\delta_1}$ we get

$$|H_{a,z}(w)| \le \delta_2 + \delta_1^2 h_2 + \delta_1^3 h_3.$$

It can be checked that $2\delta_1 h_2 + 3\delta_1^2 h_3 < 1$ and $\delta_2 + \delta_1^2 h_2 + \delta_1^3 h_3 < \delta_1$. Consequently, for each $a \in \mathcal{I}_0$ and $z \in \overline{B}_{\delta_2}$ we obtain $H_{a,z}\left(\overline{B}_{\delta_1}\right) \subseteq \overline{B}_{\delta_1}$, and $H_{a,z}$ is a contraction on $\overline{B}_{\delta_1}$. Therefore, for each $z \in \overline{B}_{\delta_2}$ there is a unique $w^*(z) \in \overline{B}_{\delta_1}$ with $h(w^*(z)) = z$, and the relation $h\left(\overline{B}_{\delta_1}\right) \supseteq \overline{B}_{\delta_2}$ follows. Define h^{-1} as $\overline{B}_{\delta_2} \ni z \mapsto w^*(z) \in \mathbb{C}$. Clearly, $h^{-1}\left(\overline{B}_{\delta_2}\right) \subseteq \overline{B}_{\delta_1}$.

In the subsequent sections it is principal to estimate the moduli of $h(w)$, $G(h(w))$ and $h^{-1}(G(h(w)))$ in (35), provided that $|w| \le \rho_0$ (see Figure 5). The magnitude of the remaining term R_2 highly depends on the size of the set, where the estimations are considered. Recall $\rho_1 = 0.02234$ from Proposition 7, and set

$$\rho_0 = 0.021, \quad \rho_2 = 0.02354, \quad \rho_3 = 0.02532. \tag{44}$$

Clearly, $\rho_0, \rho_3 \le \delta_1$ and $\rho_1, \rho_2 \le \delta_2$. Our aim is to show that

$$h(B_{\rho_0}) \subseteq B_{\rho_1}, \quad G(B_{\rho_1}) \subseteq B_{\rho_2}, \quad h^{-1}(B_{\rho_2}) \subseteq B_{\rho_3}$$

see Figure 5. For the first relation we chose ρ_0 such that $h^{max}(\rho_0) \le \rho_1$ holds. Consequently, during the study of G we can assume that $|z| \le \rho_1$. The other two relations will be shown in the subsequent subsections.

5.3 The estimation of R_1

From (32), (20) and (22) it is straightforward that in order to obtain an estimate of R_1, we need the at least sixth order terms of the real polynomial

$$\mathcal{R}_1(r, c) = r + d_0\, \tilde{g}^{max}\left(r, \phi^{max}(r) + c\, r^5\right) = \sum_{k=1}^{10} b_{1,k}(c)\, r^k,$$

since we estimated by order in $\tilde{g}^{max}$ and ϕ^{max}. Thus, using $|c| \le C_1$ and $|z| = r \le \rho_1$ we obtain

$$|R_1| \le \sum_{k=6}^{10} b_{1,k}(c)|z|^k \le \sum_{k=6}^{10} b_{1,k}(C_1)\rho_1^{k-6}|z|^6 \le 16550|z|^6,$$

i.e., $R_{10} = 16550$. Using (39) we get the following finite-order polynomial estimate of G

$$|G(z)| \le G^{max}(|z|) \tag{45}$$

with

$$G^{max}(r) = r + G_2 r^2 + G_3 r^3 + G_4 r^4 + G_5 r^5 + R_{10} r^6. \tag{46}$$

Note that the coefficients are independent of a and c, and we estimate by order in the sense that the higher order terms are estimated together in the term R_{10}. Now, it can be checked that $G^{max}(\rho_1) \le \rho_2$ holds. Hence, we get $G(B_{\rho_1}) \subseteq B_{\rho_2}$, so during the study of h^{-1} we can assume that $|z| \le \rho_2$.

5.4 The estimation of R_3 – the higher order terms in h^{-1}

Now, we turn our attention to the estimation of R_3 in (37), which consists of the sixth and higher order terms of h^{-1}. More precisely, R_3 is defined as $B_{\delta_2} \ni z \mapsto h^{-1}(z) - h_0^{-1}(z) \in \mathbb{C}$ and we need an estimate $|R_3(z)| < R_{30}|z|^6$, assuming $|z| \le \rho_2$. First, we give an estimate of type $|R_3(h(w))| \le R_{31}|w|^6$.

Set $\tilde{\rho}_3 = 0.0256$. Combining (42) with the facts that $[0, \rho_1] \ni s \mapsto s - h_2 s^2 h_3 s^3 \in \mathbb{R}$ is strictly increasing, and $\tilde{\rho}_3 - h_2\tilde{\rho}_3^2 h_3 \tilde{\rho}_3^3 > \rho_2$, we can give the a priori estimate $\tilde{\rho}_3$ of $|h^{-1}(B_{\rho_2})|$. That is, $w = h^{-1}(z)$ satisfies $|w| < \tilde{\rho}_3$, provided that $|z| < \rho_2$.

Using the definition of h_0^{-1} and h, we obtain

$$R_3(h(w)) = h^{-1}(h(w)) - h_0^{-1}(h(w)) = w - h_0^{-1}(h(w)) = \sum_{6 \le i+j \le 15} b_{3,ij} w^i \overline{w}^j,$$

where the coefficients $b_{3,ij} = b_{3,ij}(a)$ are complex. Furthermore, consider the composition

$$\mathcal{R}_3(x) = \tilde{h}_0^{max}\big(h^{max}(x)\big) = \sum_{k=1}^{15} b_{3,k}\, x^k$$

of the real functions h^{max} and $\tilde{h}_0^{max}$. Since in (40) we estimated by order, it follows that $\sum_{i+j=k} |b_{3,ij}| \le b_{3,k}$ for every $a \in \mathcal{I}_0$ and $k \in \{6, 7, \ldots, 15\}$. Thus,

$$|R_3(h(w))| \le \sum_{k=6}^{15} b_{3,k}\, |w|^k \le \sum_{k=6}^{15} b_{3,k}\, \tilde{\rho}_3^{k-6}|w|^6,$$

assuming $|w| \le \tilde{\rho}_3$. Using (43) we gain $|w| \le \eta(\tilde{\rho}_3)|z|$ and

$$|R_3(z)| \le \sum_{k=6}^{15} b_{3,k}\, \tilde{\rho}_3^{k-6}\big(\eta(\tilde{\rho}_3)\big)^6|z|^6 \le 9814|z|^6,$$

i.e., $\tilde{R}_{30} = 9814$. Consequently, we can give the estimation

$$\left|h^{-1}(z)\right| \le r + \tilde{h}_2 r^2 + \tilde{h}_3 r^3 + \tilde{h}_4 r^4 + \tilde{h}_5 r^5 + \tilde{R}_{30} r^6,$$

where $|z| = r$. However, using $|z| < \rho_2$ we get $|h^{-1}(z)| \le \rho_3$, where ρ_3 was defined in (44). It means that h^{-1} maps B_{ρ_2} actually into B_{ρ_3}, i.e., we obtain a better estimate of $|h^{-1}(B_{\rho_2})|$.

Thus, repeating the argument above with ρ_3 instead of $\tilde{\rho}_3$ we get a better estimation of the higher order terms

$$|R_3(z)| \leq \sum_{k=6}^{15} b_{3,k}\, \rho_3^{k-6} \big(\eta(\rho_3)\big)^6 |z|^6 \leq 9744|z|^6,$$

i.e., $R_{30} = 9744$. We obtain

$$\left|h^{-1}(z)\right| \leq \tilde{h}^{max}(|z|) \tag{47}$$

with

$$\tilde{h}^{max}(r) = r + \tilde{h}_2 r^2 + \tilde{h}_3 r^3 + \tilde{h}_4 r^4 + \tilde{h}_5 r^5 + R_{30} r^6. \tag{48}$$

It can be checked that $\tilde{h}^{max}(\rho_2) \leq \rho_3$ holds with R_{30}, so $h^{-1}(B_{\rho_2}) \subseteq B_{\rho_3}$. Note that the coefficients in (48) are independent of a and c, and we estimate by order in (47) in the sense that the higher order terms are estimated together in the term R_{30}.

5.5 The estimation of R_2 – the higher order terms in $h^{-1}(G(h))$

Now, we turn our attention to the estimation of R_2, which denotes the at least fourth order terms in $h^{-1}(G(h(w)))$. To obtain a better estimate we handle the fourth (R_{24}), the fifth (R_{25}) and the higher order terms (R_{26}) separately, i.e., set $R_2 = R_{24} + R_{25} + R_{26}$.

As $h^{-1}(G(h(w)))$ is in normal form, it can be written in the form

$$h^{-1}(G(h(w))) = \lambda w + c_1 w^2 \overline{w} + \sum_{4 \leq i+j \leq 5} \frac{\beta_{ij}}{i!j!} w^i \overline{w}^j + R_{26},$$

where coefficients $c_1 = c_1(a)$ and $\beta_{ij} = \beta_{ij}(a)$ are complex, and $R_{26} = R_{26}(w, \overline{w}, a, c) = O(|w|^6)$. Note that coefficients β_{ij} are independent of c, as in $T(r, C)$ the parameter c appears only in the term $c|z|^5$. Actually, that is why we chose the fourth order approximation of the center manifold and the fifth order term $c|z|^5$ in T. We can explicitly determine even the coefficients of the fourth and fifth order terms in $h^{-1}(G(h(w)))$. Thus, we can obtain better accuracy during the estimation of the higher order terms of $h^{-1}(G(h(w)))$, and consequently, we can get a larger ρ_0. The formulas of coefficients β_{ij} for $4 \leq i + j \leq 5$, and the Lyapunov-coefficient c_1 can be found in the Appendix. With interval arithmetic we get

$$\sum_{i+j=4} \frac{|\beta_{ij}|}{i!j!} \leq 33.549 \qquad \text{and} \qquad \sum_{i+j=5} \frac{|\beta_{ij}|}{i!j!} \leq 148.723. \tag{49}$$

Thus, with $R_{240} = 33.549$ and $R_{250} = 148.723$ we obtain

$$|R_{24}| \leq R_{240}|w|^4 \qquad \text{and} \qquad |R_{25}| \leq R_{250}|w|^5.$$

Using (41), (46) and (48) we gain the real polynomial

$$\mathcal{R}_2(r) = \tilde{h}^{max}\big(G^{max}\left(h^{max}(r)\right)\big) = \sum_{k=1}^{108} b_{2,k}\, r^k.$$

Since in (40), (45) and (47) we estimate by order, we get

$$|R_{26}| \leq \sum_{k=6}^{108} b_{2,k} |w|^k.$$

Using $|w| \leq \rho_0$ we obtain

$$|R_{26}| \leq \sum_{k=6}^{108} b_{2,k} \rho_0^{k-4} |w|^4 \leq 35.122|w|^4,$$

so $R_{260} = 35.122$.

Combining these three results we obtain

$$|R_2| \le |R_{24}| + |R_{25}| + |R_{26}| \le (R_{240} + R_{250}\rho_0 + R_{260})|w|^4 = 71.8|w|^4,$$

and consequently, $R_{20} = 71.8$.

5.6 The dynamics in direction z

Now, with our previous estimate of R_2 we can finish our proof. Since

$$\left|\lambda(a)w + c_1(a)w^2\overline{w} + R_2\right| \le |w|\left(\left||\lambda| + \tilde{c}_1|w|^2\right| + R_{20}|w|^3\right),$$

where $\tilde{c}_1 = c_1|\lambda|/\lambda$, we only need to prove

$$\left||\lambda| + \tilde{c}_1|w|^2\right| + R_{20}|w|^3 < 1 \tag{50}$$

for every $a \in \mathcal{I}_0$ and $w \in \mathbb{C}$ with $0 < |w| \le \rho_0$.

To this end, we show that with a suitable $R_4 > 0$ the inequality

$$\left||\lambda| + \tilde{c}_1|w|^2\right| \le |\lambda| + (\operatorname{Re}\tilde{c}_1)|w|^2 + R_4|w|^3$$

holds, or equivalently

$$0 \le 2R_4|\lambda| - (\operatorname{Im}\tilde{c}_1)^2|w| + 2R_4(\operatorname{Re}\tilde{c}_1)|w|^2 + R_4^2|w|^3$$

for $|w| \le \rho_0$ and $a \in \mathcal{I}_0$. With interval arithmetic we obtain that $\operatorname{Re}\tilde{c}_1$ and $\operatorname{Im}\tilde{c}_1$ are negative, $|\operatorname{Re}\tilde{c}_1| \le 1.513$, $|\operatorname{Im}\tilde{c}_1| \le 2.481$ and $|\lambda| \ge 0.9952$ for $a \in \mathcal{I}_0$. So it can be checked that $R_4 = 0.065$ is a suitable choice, assuming $|w| \le \rho_0$. Therefore, the left hand side of (50) can be written in the following form

$$\begin{aligned}\left||\lambda| + \tilde{c}_1|w|^2\right| + R_{20}|w|^3 &\le \left(|\lambda| + \operatorname{Re}\tilde{c}_1|w|^2\right) + (R_4 + R_{20})|w|^3\\ &\le 1 + \left(\operatorname{Re}\tilde{c}_1 + (R_4 + R_{20})|w|\right)|w|^2,\end{aligned}$$

which is less than 1, provided that

$$0 < |w| < \frac{-\operatorname{Re}\tilde{c}_1}{R_4 + R_{20}}.$$

Using the fact that $|\operatorname{Re}\tilde{c}_1| \ge 1.511$ we obtain

$$\rho_0 < \frac{-\operatorname{Re}\tilde{c}_1}{R_4 + R_{20}}.$$

Therefore, the inequality (50) holds for all $w \in \mathbb{C}$ with $0 < |w| \le \rho_0$. Consequently, with the arbitrarily chosen $a \in \mathcal{I}_0$ and $|c| \le C_1$ we proved (38) for all $w \neq 0$ with $|w| \le \rho_0$.

Note that we obtained also that the bifurcation is supercritical since $\operatorname{Re}\tilde{c}_1(a_0)$ is negative.

5.7 Combining the y- and z-directional dynamics

Now, let $a \in \mathcal{I}_0$ be fixed and consider the set

$$T' = \left\{(z, y) \in \mathbb{C} \times \mathbb{R} :\ \left|h^{-1}(z)\right| \le \rho_0,\ |y - \phi(z)| \le C_1|z|^5\right\}.$$

Note that ρ_0, h and h^{-1} are independent of c. Clearly, $T' \subseteq T(\rho_1, C_1)$, since $h^{max}(\rho_0) \le \rho_1$. Considering an arbitrary point $(z_0, y_0) \in T'$ there exists a $c \in \mathbb{R}$ with $|c| \le C_1$ such that

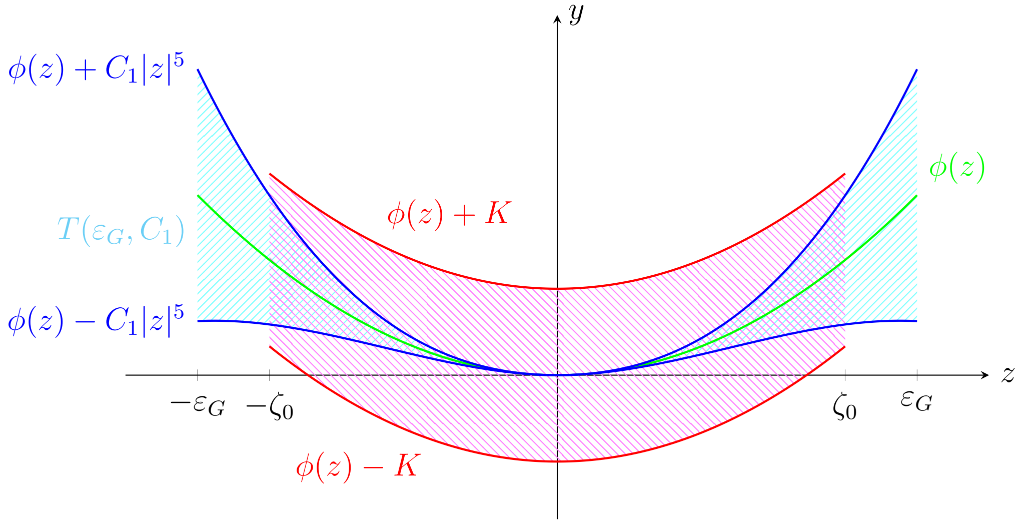


Figure 6. The set $\tilde{T}(r, K)$

$(z_0, y_0) \in \tau(\rho_1, c)$. For $(z_1, y_1) = H_a(z_0, y_0)$ the inequality $|h^{-1}(z_1)| \le \rho_0$ follows from (38). For the y-directional dynamics we obtain $|y_1 - \phi(z_1)| \le C_1|z_1|^5$ from Proposition 10. Combining these two results we obtain that T' is invariant under (14). Consequently, $(z_n, y_n) \in T'$ for every $n \in \mathbb{N}_0$, where $(z_{n+1}, y_{n+1}) = H(z_n, y_n)$.

Introduce $w_n = h^{-1}(z_n)$ for $n \in \mathbb{N}_0$. Combining (38) with the fact that (z_n, y_n) remains in T' for every $n \in \mathbb{N}_0$, we get that $(|w_n|)_{n=0}^{\infty}$ is strictly decreasing. From the continuity of functions h, G and h^{-1} we obtain $w_n \to 0$, consequently, $z_n \to 0$ also holds. Therefore, T' is in the region of attraction of the trivial fixed point of map (14). Note that there exists a $\hat{c}$ with $|\hat{c}| \le C_1$ such that $(z_1, y_1) \in \tau(\rho_1, \hat{c})$. It does not cause any trouble if $c \ne \hat{c}$, i.e., if the point (z_0, y_0) jumps from $\tau(\rho_1, c)$ to $\tau(\rho_1, \hat{c})$, as maps h and h^{-1} are the same for every $|c| \le C_1$.

Finally, ε_G was chosen such that $\tilde{h}^{max}(\varepsilon_G) \le \rho_0$, so $T(\varepsilon_G, C_1) \subseteq T'$ is also in the region of attraction of the fixed point. Although $T(\varepsilon_G, C_1)$ is not invariant, its projection on the z-coordinate is a disc, so it is easier to work with it in the computer-assisted part of the proof. Since a was an arbitrary value from $\mathcal{I}_0$, the proof of Proposition 11 is complete. $\square$

Notice that $T(\varepsilon_G, C_1)$ is not a neighborhood of the origin since the y-directional thickness is zero for $z = 0$. In the following section we construct thicker sets around the origin to obtain a proper neighborhood of it.

6 Constructing the attracting neighborhood

In the previous sections we showed that $T(\varepsilon_G, C_1)$ is in the region of attraction of the origin. Based on this set we construct attracting neighborhoods around the fixed point of (14).

Introduce the set (see Figure 6)

$$\tilde{T}(r, K) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \le r, \ |\phi(z) - y| \le K\}. \tag{51}$$

Notice that in $\tilde{T}(r, K)$, unlike in $T(r, C)$, the y-directional thickness of the set is independent of the z-coordinate. Using Proposition 9 we can construct an attracting neighborhood around the fixed point.

Proposition 12. *Let $\zeta_0 = 0.01883$ and $K_0 = 2.74 \cdot 10^{-5}$. The set $\tilde{T}_0 = \tilde{T}(\zeta_0, K_0)$ is in the region of attraction of the fixed point of* (14).

Proof. Let $(z_0, y_0) \in \tilde{T}_0$ and $(z_1, y_1) = H_a(z_0, y_0)$. The value ζ_0 was chosen such that

$$\zeta_0 + d_0 \tilde{g}^{max}\big(\zeta_0, \phi^{max}(\zeta_0) + K_0\big) \le \varepsilon_G,$$

and therefore, $|z_1| \le \varepsilon_G$ also holds. Recall that in Proposition 9 the constant σ was chosen such that it satisfies $\phi(\zeta_0) + K_0 \le \sigma$. Clearly, $\zeta_0 < \rho_1$, and we can use Proposition 9 to obtain

$$|k_1| \le L|k_0| + C_0|z_0|^5 \le LK_0 + C_0|z_0|^5,$$

where $k_i = y_i - \phi(z_i)$ for $i \in \{0, 1\}$.

We chose K_0 such that $K_0 \le (3/2)C_1\zeta_0^5$ holds, so we gain that $|z_1| > \zeta_0$ implies

$$|k_1| \le LK_0 + C_0\zeta_0^5 \le \left(\frac{3}{2}LC_1 + C_0\right)\zeta_0^5 \le C_1\zeta_0^5 \le C_1|z_1|^5.$$

Hence, $(z_1, y_1) \in T(\varepsilon_G, C_1)$, i.e., (z_0, y_0) is in the region of attraction of the fixed point.

If $|z_1| \le \zeta_0$, then it is enough to examine the case $(z_0, y_0) \notin T(\varepsilon_G, C_1)$, otherwise, (z_0, y_0) is clearly inside the region of attraction. Therefore, suppose $|k_0| \ge C_1 z_0^5$. We get

$$|k_1| \le L|k_0| + C_0|z_0|^5 \le \left(L + \frac{C_0}{C_1}\right)|k_0| \le \frac{2}{3}K_0, \tag{52}$$

i.e., $(z_1, y_1) \in \tilde{T}\left(\zeta_0, (2/3)K_0\right) \subseteq \tilde{T}_0$. Repeating the argument above, we obtain that as long as $(z_n, y_n)_{n=0}^{\infty}$ is outside of $T(\varepsilon_G, C_1)$, the sequence exponentially decays to the approximation $y = \phi(z)$ of the center manifold. Consequently, either there is an $n_0 \in \mathbb{N}$ such that $(z_{n_0}, y_{n_0}) \in T(\varepsilon_G, C_1)$, or $(z_n, y_n) \in \tilde{T}\left(\zeta_0, (2/3)^n K_0\right) \setminus T(\varepsilon_G, C_1)$ holds for every $n \in \mathbb{N}$. It is easy to see that $\tilde{T}\left(\zeta_0, (2/3)^n K_0\right) \setminus T(\varepsilon_G, C_1)$ shrinks to the origin, as n tends to infinity. Thus, (z_n, y_n) tends to the fixed point also in this case. Consequently, $\tilde{T}_0$ is also in the region of attraction of the fixed point. □

Now, $\tilde{T}_0$ is a neighborhood of the origin. However, during the computer-aided part of our method we need thicker (in the direction y) neighborhoods.

Proposition 13. *Let $K_n = (3/2)^n K_0$ and*

$$\begin{array}{lllll}\zeta_1 = 0.01804, & \zeta_2 = 0.01731, & \zeta_3 = 0.01664, & \zeta_4 = 0.01601, & \zeta_5 = 0.01543, \\ \zeta_6 = 0.01489, & \zeta_7 = 0.01437, & \zeta_8 = 0.01389, & \zeta_9 = 0.01342, & \zeta_{10} = 0.01297.\end{array}$$

Then $\tilde{T}_n = \tilde{T}\left(\zeta_n, K_n\right)$ is in the region of attraction of the origin for $n \in \{1, 2, \ldots, 10\}$.

Proof. We only need to show that $H_a(\tilde{T}_n \setminus \tilde{T}_{n-1}) \subseteq \tilde{T}_{n-1}$ holds for every $n \in \{1, 2, \ldots, 10\}$. It can be checked that the decreasing sequence $(\zeta_n)_{n=0}^{10}$ satisfies

$$\zeta_n + d_0 g^{max}\big(\zeta_n, \phi^{max}(\zeta_n) + K_n\big) \le \zeta_{n-1},$$

so the z-direction inclusion is shown. Proposition 9 can be applied, since $\zeta_n \le \rho_1$ for every $n \in \{1, 2, \ldots, 10\}$, and σ was chosen such that $\phi(\zeta_0) + (3/2)^{10} K_0 \le \sigma$. Hence, the y-directional inclusion follows from (52). □

Note that for every $n \in \{0, 1, \ldots, 10\}$ the set $\tilde{T}_n$ is an attracting neighborhood of the trivial fixed point of (14), and the size of these neighborhoods are independent of $a \in \mathcal{I}_0$. Clearly, it still needs to be transformed, in order to obtain a neighborhood $\mathcal{M}$ of u_A. This transformation is, however, handled by the algorithm, see Section 8.

7 Computer-assisted part for a fixed a

To study the global behavior of map (2) we follow the method from [3] and [4]. In this section we show how use it for a fixed $a \in [4/3, a_0]$. The method is essentially the same as in [3], so we just outline the steps. The only main difference is that the 2-dimensional squares are replaced by 3-dimensional cubes. The detailed description of the method along with the correctness of that can be found in [3] and in [4].

For a given $a \in [4/3, a_0]$ we associate the map (2) with a directed graph reflecting the behavior of the map up to a given resolution. Using this graph we show that every point of S enters the previously obtained attracting neighborhood $\mathcal{M}$ of the nontrivial fixed point u_A.

Let D be a subset of $\mathbb{R}^n$. A set $\mathfrak{S}$ is called a *cover* of D if the elements of $\mathfrak{S}$ are subsets of $\mathbb{R}^n$ and $\cup_{\mathfrak{s}\in\mathfrak{S}}\mathfrak{s} \supset D$. Let a map $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^n$, a subset $D \subseteq D_f$ and a cover $\mathfrak{S}$ of D be given. The directed graph $G(V, E)$ is called a *graph representation* of f on D with respect to $\mathfrak{S}$ if there exists a bijection $\iota : V \to \mathfrak{S}$ such that

$$f(\iota(v_1) \cap D) \cap \iota(v_2) \cap D \neq \emptyset \Rightarrow (v_1, v_2) \in E \tag{53}$$

for all $v_1, v_2 \in V$. Therefore, if $x \in \mathfrak{s}_1 \in \mathfrak{S}$ and $y = f(x) \in \mathfrak{s}_2 \in \mathfrak{S}$, then $(v_1, v_2) \in E$ with $v_1 = \iota^{-1}(\mathfrak{s}_1) \in V$ and $v_2 = \iota^{-1}(\mathfrak{s}_2) \in V$. The reverse implication in (53) is not necessarily true, namely, if $(v_1, v_2) \in E$, it is not sure that there exists $x \in \iota(v_1)$ with $f(x) \in \iota(v_2)$. So the graph representation can be regarded as some kind of upper estimate of the original map f. If we would like to determine the possible location of $f(x)$ for a given $x \in D$, we can do it with the help of the graph, since during the iteration of f a point can move forward only along the edges. In the following we take the liberty to handle the elements of the cover as vertices and vice versa, omitting the use of ι.

The construction of the graph representation in our case is the following. For a fixed $k \in \mathbb{N}$ we divide the unit cube $[0, 1]^3$ parallel to the faces into small congruent closed cubes with side length $r = 2^{-k}$. These small cubes serve as the cover $\mathfrak{S}$ of S and also as the vertices of the graph, too. To determine the edges we construct with interval arithmetic methods (see [15]) a rectangular cuboid for every small cube $\mathfrak{s}_1$ such that this cuboid contains $f(\mathfrak{s}_1)$. So there is an edge from $\mathfrak{s}_1$ to $\mathfrak{s}_2$ if the cuboid constructed for $\mathfrak{s}_1$ intersects the small cube $\mathfrak{s}_2$. Note that instead of map (2) we use the third iterate of it, since the formula is still compact enough not to cause big overestimates in interval arithmetic and it considerably speeds up the calculations.

A graph is *strongly connected* if there are v_1v_2 and v_2v_1 (directed) paths for every $v_1 \neq v_2$ vertices of the the graph. We use the following decomposition of a directed graph (see [16]).

Proposition 14. *The vertices of a directed graph can be classified and the classes can be ordered such that*

- *the subgraphs spanned by the classes are strongly connected, and*
- *for every directed edge between these classes, the class of the tail of this edge precedes the class of the head of it,*

moreover, the partition above is unique.

The aforementioned classes are called the *strongly connected components* (*SCC*) of the graph. A strongly connected component is called *nonessential* if it consists of one vertex without loop. Otherwise, we call it *essential.*

From the graph representation and from Proposition 14 it is clear what happens to an arbitrary point of S during the iteration of f. Starting from a small cube containing this point it moves to an other (possibly the same) small cube along a directed edge. If we are not in an essential SCC we step out of this small cube not returning to here afterwards because of the ordering of the SCCs. If we are in an essential SCC it can happen that the point stays here forever, or the point steps out of this SCC, but in this case it can not return to this SCC any more. Since the graph is finite, it is straightforward that for every point of S there exists an essential strongly connected component, which the point enter during the iteration and never leaves it. Therefore, we only need to study the essential SCCs. Our aim is to show that the SCCs are in the attracting neighborhood $\mathcal{M}$ of the fixed point u_A, which neighborhood was constructed analytically in the previous sections.

It is important to note that it is possible for some essential SCC that in fact none of the points of S can get stuck here. Actually, this can be the case close to the trivial fixed point $(0, 0, 0)$ of map (2). Since this fixed point is a saddle, the small cube containing this point always has a loop in the graph representation, consequently, it is always an element of an essential SCC. Let us suppose that this SCC is inside the set $[0, \tilde{A}]^3$ for some $\tilde{A} < A$, and a point gets stuck in this SCC, i.e., $(u^{3k})_{k=0}^{\infty}$ is in $[0, \tilde{A}]^3$, (since we considered the third iterate of f). From the transition graph (see Figure 3) it follows that $(u^k)_{k=0}^{\infty}$ is also in $[0, \tilde{A}]^3$. Using Proposition 4 we obtain $u^k \to u_A$, which contradicts $\tilde{A} < A$. Consequently, if the essential SCC containing the origin is included in $[0, \tilde{A}]^3$ for some $\tilde{A} < A$, then we can exclude this SCC from our study.

As a next step we refine the partition by dividing the small cubes into eight identical smaller ones. Then we restart the process, i.e., we determine their images with reliable numerical methods and construct the SCCs again. Note that with the refinements the graph representation becomes a more and more accurate approximation of the represented map, so it is likely to appear new nonessential SCCs, which can be excluded from our study. This property slows a little bit down the eightfold increase in the number of vertices caused by the refinement. Finally, instead of checking after every refinement, whether the remaining SCCs are in the analytically constructed attracting neighborhood $\mathcal{M}$, we can simply remove the small cubes lying entirely in $\mathcal{M}$. So for a fixed a the main theorem is proven if the set of the essential SCCs is empty after some refinements. For the correctness of these steps see [3]. Note that in Propositions 6 and 13 we obtained attracting neighborhoods in $\mathbb{C} \times \mathbb{R}$. Therefore, we need a transformation so that we can determine whether a small cube is inside the region of attraction of the fixed point or not.

Algorithm Proving the global stability of u_A for the logistic map

procedure Log3d
    $V \leftarrow$ the initial partition $r = 2^{-10}$
    repeat
        $E \leftarrow$ construct the edges with reliable numerical method
        $C \leftarrow$ determine the SCCs of directed graph $G(V, E)$
        remove vertices of the nonessential SCCs from V
        remove vertices of the SCC at the origin from V **if** possible
        remove the initial attracting neighborhood from V
        $V \leftarrow$ refine the partition $r \leftarrow r/2$
    until $|V| = \emptyset$
end procedure

We also mention here the fact that in $T(r, C)$ and consequently in $\tilde{T}(r, K)$ the constant r can be enlarged only at the expense of C (and K), so we need to find the balance between them. In the optimal case the obtained attracting neighborhood should resemble the shape of the remaining set after some iteration in the computer-aided part of the method. Neither r, nor C can be too small. If r is not sufficiently large, we need more refinement steps and the exponentially growing number of vertices causes some difficulties. On the other side, if C (and K) is too small, then the initial attracting neighborhood is too thin in direction y, and

interval	Subinterval length	Proposition	Running time
$[4/3, 1.56]$	10^{-3}	6	255 sec
$[1.56, 1.6]$	10^{-4}	6	35 min
$[1.6, a_0 - 10^{-2}]$	10^{-5}	6	4 h
$[a_0 - 10^{-2}, 1.616]$	10^{-4}	13	52 min
$[1.616, a_0]$	10^{-5}	13	10 h

Table 1. The partition of the parameter range

we can not remove any small cubes at all for a long time, as the small cubes are larger than the thickness of the attracting neighborhood. Here, we remark that the removal of the cubes in the initial attracting neighborhood considerably reduces the number of the vertices, which results in faster running of the algorithm.

The program code and the outputs can be found on [17]. Note that the aforementioned method can be regarded as a proof, since the graph problems are finite, so the computer can work on them punctually. Moreover, the method used during the construction of edges was executed with reliable numerical methods. Thus, if we would have sufficient time, then we could reconstruct by hand the parts which were executed by computer, and we would come to the same conclusion, if our estimates are as good as the computer's.

8 Computer-assisted part for an interval of a

In the previous sections we obtained an attracting neighborhood and then a method to prove the global stability of the nontrivial fixed point for a fixed $a \in [4/3, a_0]$. In this section we show how to modify our method to handle not only a single value of the interval $[4/3, a_0]$ but also a small subinterval $[a] = [a_-, a_+]$ of that, instead.

When we replace the single parameter value with an interval, we obtain rougher estimates, as we handle more parameter values together at the same time. Far away from a_0 the convergence is relatively fast, so we can use longer subintervals when we divide the interval $[4/3, a_0]$ into small intervals (see Table 1). Far away from a_0 the algorithm is still fast enough with these rougher estimates. However, close to a_0 the convergence is much slower, so the precision of estimates is more crucial in this case. Therefore, we need to use finer partition close to a_0.

When we apply our method to a small subinterval $[a]$, essentially two modifications need to be done. First, we need to adjust the function (2) during the construction of edges in the graph representation. For a given subinterval $[a]$ and a given small cube $\mathfrak{s}$ we consider a set of small cubes such that they cover $f_a^3(\mathfrak{s})$ for every $a \in [a]$. Second, we also need to modify the attracting neighborhood we remove during the algorithm. For a given subinterval $[a]$ the attracting neighborhood must be chosen such that it is inside the region of attraction of the fixed point for every $a \in [a]$. Note that not only the size of the neighborhood but also the location of the fixed point u_A can vary for different values from $[a]$.

For $[a] \subseteq [4/3, a_0 - 10^{-2}]$ we use the linearized map and Proposition 6. For a given $[a]$ the size of the neighborhood can be chosen as $\min_{a\in[a]} \xi(a)$. For $[a] \subseteq \mathcal{I}_0$ we use the attracting neighborhoods from Proposition 13 which are obtained by the center manifold reduction and the bifurcational normal form. In this case the size of the neighborhood is independent of the choice of $[a]$. In both propositions the neighborhood is given in the $\mathbb{C} \times \mathbb{R}$, but the small cubes are in $\mathbb{R}^3$, and thus we need to transform them first. We accomplish the transformation with computer using interval arithmetic calculation.

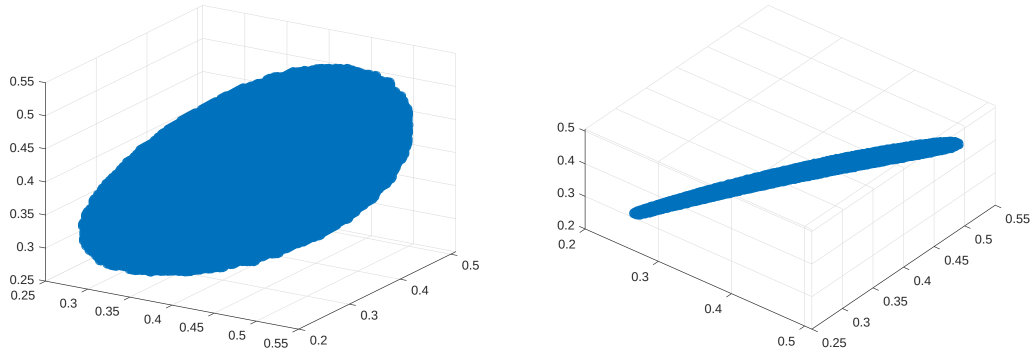


Figure 7. The output from two different points of view for $a = 1.612$ after 4 iteration

For a given $[a]$ and small cube k we use interval arithmetic calculations to determine the new coordinates in the z–y space. First, we need to shift the small cube with the interval version $u_{[A]}$ of u_A. Here, every coordinate of $u_{[A]}$ is an interval containing u_A for every $a \in [a]$, i.e., every coordinate of $u_{[A]}$ is $[A] = [1 - 1/a_-, 1 - 1/a_+]$. Then similarly, we apply the interval version $Q^{-1}_{[a]}$ of (11) to obtain $[y]$ and $[z]$. Here, every element of Q^{-1}_a is replaced by an interval containing that element of Q^{-1}_a for every $a \in [a]$. Thus, $[y] \subseteq \mathbb{R}$ is an interval and $[z] \subseteq \mathbb{C}$ is a disc such that

$$Q^{-1}_a(u - u_A) \in \{(z, y) : \ z \in [z],\, y \in [y]\}$$

for every $a \in [a]$ and $u \in \mathfrak{s}$.

For $[a] \subseteq [4/3, a_0 - 10^{-2}]$ we only need to check whether

$$\max_{y \in [y]} |y| + \max_{z \in [z]} |z| \le \min_{a \in [a]} \xi(a).$$

For $[a] \subseteq \mathcal{I}_0$ first, we need to determine with interval arithmetic the image $[\phi] \subseteq \mathbb{R}$ of $[z]$ under the map $\phi_{[a]}$. Here, $\phi_{[a]}$ means the interval version of ϕ, i.e., every coefficient ω_{ij} is replaced by a disk in the complex plane such that this disk contains $\omega_{ij}(a)$ for every $a \in [a]$. Hence, for every $a \in [a]$ and $z \in [z]$ we have $\phi_a(z) \in [\phi]$. Then we need to check the inequalities

$$\max_{z \in [z]} |z| \le \zeta_n, \qquad \max_{x \in [\phi]} |x| - \min_{y \in [y]} |y| \le K_n, \qquad \max_{y \in [y]} |y| - \min_{x \in [\phi]} |x| \le K_n$$

for some $n \in \{0, 1, \dots, 10\}$. Practically, for a given $[a]$ we use larger n at the beginning of the algorithm, since K_n needs to be large enough compared to the refinement of the partition of the unit cube. In this way we can remove a lot of small cubes close to the nontrivial fixed point. Thus, we can reduce the size of the graph which considerably speeds up our algorithm. Later, when the partition is finer, we can use smaller n to obtain a larger (along the z-coordinate) attracting set, which makes our program finish earlier.

The running times can also be found in Table 1. It can be observed that close to $a_0 - 10^{-2}$ the first method using the linearization becomes less and less efficient. If our aim had been to reduce the running time, then we could have repeated the second method using the normal form and the center manifold on a larger interval. The calculations in the proofs would have differed only in the specific values. For the sake of example some results can be found on our website see [17].

The program runs successfully, so Theorem 1 is proven.

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10 Appendix

In most cases, for the sake of transparency we do not sign that coefficients depend on a, but keep it mind they actually do. Including, but not limited to $\lambda = \lambda(a)$, $\nu = \nu(a)$, $d = d(a)$, $e = e(a)$, $g_{ijk} = g_{ijk}(a)$ etc.

10.1 The eigenvalues

The complex eigenvalue with positive real part and the real eigenvalue are

$$\lambda(a) = \frac{1}{3} + \frac{1 - i\sqrt{3}}{3 \cdot 2^{\frac{2}{3}} \left(-29 + 27a + 3\sqrt{3\nabla(a)}\right)^{\frac{1}{3}}} + \frac{(1 + i\sqrt{3}) \left(-29 + 27a + 3\sqrt{3\nabla(a)}\right)^{\frac{1}{3}}}{6 \cdot 2^{\frac{1}{3}}},$$

$$\nu(a) = \frac{1}{3} - \frac{2^{\frac{1}{3}}}{3 \left(-29 + 27a + 3\sqrt{3\nabla(a)}\right)^{\frac{1}{3}}} - \frac{\left(-29 + 27a + 3\sqrt{3\nabla(a)}\right)^{\frac{1}{3}}}{3 \cdot 2^{\frac{1}{3}}},$$

where $\nabla(a) = 31 - 58a + 27a^2$.

10.2 The coefficients of $g(z, y)$

If $a > 31/27$, then

$$g_{ijk} = -a \left(i\lambda^2 + j\bar{\lambda}^2 + k\nu^2\right),$$

for $i + j + k = 2$ and $g_{ijk} = 0$ for $i + j + k > 2$.

10.3 The coefficients of the center manifold

$$\omega_{ji} = \overline{\omega}_{ij}, \qquad \omega_{20} = \frac{g_{200}e}{\lambda^2 - \nu}, \qquad \omega_{11} = \frac{g_{110}e}{\lambda\bar{\lambda} - \nu}$$

$$\omega_{30} = \frac{1}{\lambda^3 - \nu}\left(e\left(3g_{101}\omega_{20} + g_{300}\right) - 3\lambda g_{200}(\bar{d}\omega_{11} + d\omega_{20})\right)$$

$$\begin{aligned}\omega_{21} = \frac{1}{\lambda^2\bar{\lambda} - \nu}\Big(& e\left(g_{011}\omega_{20} + 2g_{101}\omega_{11} + g_{210}\right) - \bar{d}\left(\omega_{02}g_{200}\bar{\lambda} + 2\lambda\omega_{11}g_{110}\right) \\ & - d\left(\omega_{11}g_{200}\bar{\lambda} + 2\lambda g_{110}\omega_{20}\right)\Big)\end{aligned}$$

$$\begin{aligned}\omega_{40} = \frac{1}{\lambda^4 - \nu}\Big(& e(3g_{002}\omega_{20}^2 + 4g_{101}\omega_{30} + 6\omega_{20}g_{201} + g_{400}) - 3\bar{d}^2\omega_{02}g_{200}^2 \\ & - 2\bar{d}\left(2\lambda\omega_{11}\left(3g_{101}\omega_{20} + g_{300}\right) + 3d\omega_{11}g_{200}^2 + 3\lambda^2 g_{200}\omega_{21}\right) \\ & - d\left(4\lambda\omega_{20}\left(3g_{101}\omega_{20} + g_{300}\right) + 3d\omega_{20}g_{200}^2 + 6\lambda^2 g_{200}\omega_{30}\right)\Big)\end{aligned}$$

$$\omega_{31} = \frac{-1}{\lambda^3\bar{\lambda}-\nu}\Big(-e\Big(3\omega_{11}\left(g_{002}\omega_{20}+g_{201}\right)+g_{011}\omega_{30}+3g_{101}\omega_{21}+3g_{111}\omega_{20}+g_{310}\Big)$$
$$+d\Big(3\lambda\left(g_{011}\omega_{20}^2+\omega_{20}\left(2g_{101}\omega_{11}+g_{210}\right)+g_{200}\omega_{21}\bar{\lambda}\right)+\omega_{11}\bar{\lambda}\left(3g_{101}\omega_{20}+g_{300}\right)$$
$$+3dg_{110}\omega_{20}g_{200}+3\lambda^2g_{110}\omega_{30}\Big)+\bar{d}\Big(3\lambda\left(\omega_{11}\left(g_{011}\omega_{20}+g_{210}\right)+2g_{101}\omega_{11}^2+\omega_{12}g_{200}\bar{\lambda}\right)$$
$$+\omega_{02}\bar{\lambda}\left(3g_{101}\omega_{20}+g_{300}\right)+6d\omega_{11}g_{110}g_{200}+3\lambda^2g_{110}\omega_{21}+3\bar{d}\omega_{02}g_{110}g_{200}\Big)\Big)$$

$$\omega_{22} = \frac{-1}{\lambda^2\bar{\lambda}^2-\nu}\Big(-e\Big(\omega_{02}\left(g_{002}\omega_{20}+g_{201}\right)+2g_{002}\omega_{11}^2+2g_{011}\omega_{21}+g_{021}\omega_{20}+2g_{101}\omega_{12}$$
$$+4\omega_{11}g_{111}+g_{220}\Big)+d\Big(2\lambda(2g_{011}\omega_{11}\omega_{20}+\omega_{02}g_{101}\omega_{20}+2g_{110}\omega_{21}\bar{\lambda}+g_{120}\omega_{20})$$
$$+2\omega_{11}\bar{\lambda}\left(g_{011}\omega_{20}+2g_{101}\omega_{11}+g_{210}\right)+d\omega_{20}(g_{020}g_{200}+2g_{110}^2)+\lambda^2g_{020}\omega_{30}+\omega_{12}g_{200}\bar{\lambda}^2\Big)$$
$$+\bar{d}\Big(2\lambda\left(2g_{011}\omega_{11}^2+\omega_{11}\left(\omega_{02}g_{101}+g_{120}\right)+2g_{110}\omega_{12}\bar{\lambda}\right)+2\omega_{02}\bar{\lambda}\left(g_{011}\omega_{20}+2g_{101}\omega_{11}+g_{210}\right)$$
$$+2d\omega_{11}\left(g_{020}g_{200}+2g_{110}^2\right)+\lambda^2g_{020}\omega_{21}+\omega_{03}g_{200}\bar{\lambda}^2+\bar{d}\omega_{02}\left(g_{020}g_{200}+2g_{110}^2\right)\Big)\Big)$$

10.4 The fifth order terms of $\mathcal{N}(\phi(z))$

$$N_{ij} = \overline{N}_{ji}$$

$$N_{50} = 5\Big(3\bar{d}^2g_{200}(g_{200}\lambda\omega_{12}+2g_{101}\omega_{02}\omega_{20})+6d\bar{d}g_{200}(2g_{101}\omega_{11}\omega_{20}+g_{200}\lambda\omega_{21})$$
$$+3d^2g_{200}(2g_{101}\omega_{20}^2+g_{200}\lambda\omega_{30})-e(2g_{002}\omega_{20}\omega_{30}+g_{101}\omega_{40})$$
$$+\bar{d}\lambda(3g_{002}\omega_{11}\omega_{20}^2+6g_{101}\lambda\omega_{20}\omega_{21}+4g_{101}\omega_{11}\omega_{30}+2g_{200}\lambda^2\omega_{31})$$
$$+d\lambda\Big(3g_{002}\omega_{20}^3+2g_{101}(2+3\lambda)\omega_{20}\omega_{30}+2g_{200}\lambda^2\omega_{40}\Big)\Big)$$

$$N_{41} = 3\bar{d}^2\big(g_{200}^2\bar{\lambda}\omega_{03}+4g_{101}g_{110}\omega_{02}\omega_{20}+2g_{200}(2g_{101}\omega_{02}\omega_{11}+2g_{110}\lambda\omega_{12}+g_{011}\omega_{02}\omega_{20})\big)$$
$$+3d^2\big(4g_{101}\omega_{20}(g_{200}\omega_{11}+g_{110}\omega_{20})+g_{200}(2g_{011}\omega_{20}^2+g_{200}\bar{\lambda}\omega_{21}+4g_{110}\lambda\omega_{30})\big)$$
$$+\bar{d}\Big(3g_{002}\omega_{20}(4\lambda\omega_{11}^2+\bar{\lambda}\omega_{02}\omega_{20})$$
$$+6d\big(4g_{101}\omega_{11}(g_{200}\omega_{11}+g_{110}\omega_{20})+g_{200}(g_{200}\bar{\lambda}\omega_{12}+2g_{011}\omega_{11}\omega_{20}+4g_{110}\lambda\omega_{21})\big)$$
$$+4g_{101}(3\lambda^2\omega_{11}\omega_{21}+3\lambda(\bar{\lambda}\omega_{12}\omega_{20}+\omega_{11}\omega_{21})+\bar{\lambda}\omega_{02}\omega_{30})$$
$$+2\lambda(3g_{011}\lambda\omega_{20}\omega_{21}+3g_{200}\lambda\bar{\lambda}\omega_{22}+2g_{011}\omega_{11}\omega_{30}+2g_{110}\lambda^2\omega_{31})\Big)$$
$$-e(6g_{002}\omega_{20}\omega_{21}+4g_{002}\omega_{11}\omega_{30}+4g_{101}\omega_{31}+g_{011}\omega_{40})$$
$$+d\Big(3g_{002}(4\lambda+\bar{\lambda})\omega_{11}\omega_{20}^2+4g_{101}\big(3\lambda(1+\bar{\lambda})\omega_{20}\omega_{21}+3\lambda^2\omega_{11}\omega_{30}+\bar{\lambda}\omega_{11}\omega_{30}\big)$$
$$+2\lambda\big(g_{011}(2+3\lambda)\omega_{20}\omega_{30}+\lambda(3g_{200}\bar{\lambda}\omega_{31}+2g_{110}\lambda\omega_{40})\big)\Big)$$

$$
\begin{aligned}
N_{32} =\ & 3\bar{d}^2\big(2g_{110}g_{200}\bar{\lambda}\omega_{03} + 2g_{011}g_{200}\omega_{02}\omega_{11} + 2g_{110}^2\lambda\omega_{12} + g_{020}g_{200}\lambda\omega_{12} \\
& + 2g_{011}g_{110}\omega_{02}\omega_{20} + g_{101}\omega_{02}(g_{200}\omega_{02} + 4g_{110}\omega_{11} + g_{020}\omega_{20})\big) \\
& + 3d^2\big(g_{101}\omega_{20}(g_{200}\omega_{02} + 4g_{110}\omega_{11} + g_{020}\omega_{20}) + 2g_{011}\omega_{20}(g_{200}\omega_{11} + g_{110}\omega_{20}) \\
& + 2g_{110}g_{200}\bar{\lambda}\omega_{21} + 2g_{110}^2\lambda\omega_{30} + g_{020}g_{200}\lambda\omega_{30}\big) \\
& - e\big(3g_{101}\omega_{22} + g_{002}(3\omega_{12}\omega_{20} + 6\omega_{11}\omega_{21} + \omega_{02}\omega_{30}) + 2g_{011}\omega_{31}\big) \\
& + \bar{d}\Big(6g_{002}\lambda\omega_{11}^3 + 6g_{101}\lambda\omega_{11}\omega_{12} + 12g_{101}\lambda\bar{\lambda}\omega_{11}\omega_{12} + 3g_{200}\lambda\bar{\lambda}^2\omega_{13} + 3g_{101}\bar{\lambda}^2\omega_{03}\omega_{20} \\
& + 3g_{002}\lambda\omega_{02}\omega_{11}\omega_{20} + 6g_{002}\bar{\lambda}\omega_{02}\omega_{11}\omega_{20} + 6g_{011}\lambda\bar{\lambda}\omega_{12}\omega_{20} + 3g_{101}\lambda^2\omega_{02}\omega_{21} \\
& + 6g_{101}\bar{\lambda}\omega_{02}\omega_{21} + 6g_{011}\lambda\omega_{11}\omega_{21} + 6g_{011}\lambda^2\omega_{11}\omega_{21} + 6d\big(2g_{110}g_{200}\bar{\lambda}\omega_{12} \\
& + g_{101}\omega_{11}(g_{200}\omega_{02} + 4g_{110}\omega_{11} + g_{020}\omega_{20}) + 2g_{011}\omega_{11}(g_{200}\omega_{11} + g_{110}\omega_{20}) + 2g_{110}^2\lambda\omega_{21} \\
& + g_{020}g_{200}\lambda\omega_{21}\big) + 6g_{110}\lambda^2\bar{\lambda}\omega_{22} + 2g_{011}\bar{\lambda}\omega_{02}\omega_{30} + g_{020}\lambda^3\omega_{31}\Big) \\
& + d\Big(3g_{002}\omega_{20}(2\lambda\omega_{11}^2 + 2\bar{\lambda}\omega_{11}^2 + \lambda\omega_{02}\omega_{20}) + 6g_{011}\lambda\omega_{20}\omega_{21} \\
& + 6g_{011}\lambda\bar{\lambda}\omega_{20}\omega_{21} + 3g_{200}\lambda\bar{\lambda}^2\omega_{22} + 6g_{011}\lambda^2\omega_{11}\omega_{30} + 2g_{011}\bar{\lambda}\omega_{11}\omega_{30} \\
& + 3g_{101}(2\lambda\omega_{12}\omega_{20} + \bar{\lambda}^2\omega_{12}\omega_{20} + 2\bar{\lambda}\omega_{11}\omega_{21} + 4\lambda\bar{\lambda}\omega_{11}\omega_{21} + \lambda^2\omega_{02}\omega_{30}) \\
& + 6g_{110}\lambda^2\bar{\lambda}\omega_{31} + g_{020}\lambda^3\omega_{40}\Big)
\end{aligned}
$$

10.5 The lower order terms of $G(z)$

$$\bar{d}\,G_{ij} = d\,\bar{G}_{ji}$$

$$
\begin{aligned}
G_{20} &= -\,2ad\lambda^2, \qquad G_{11} = -ad(\lambda^2 + \bar{\lambda}^2), & G_{30} &= -\,6ad(\lambda^2 + \nu^2)\omega_{20}, \\
G_{21} &= -\,ad\big(2\lambda^2\omega_{11} + 2\nu^2\omega_{11} + \bar{\lambda}^2\omega_{20} + \nu^2\omega_{20}\big), & G_{40} &= -\,ad(6\nu^2\omega_{20}^2 + 4\lambda^2\omega_{30} + 4\nu^2\omega_{30})
\end{aligned}
$$

$$
\begin{aligned}
G_{31} &= -\,ad(6\nu^2\omega_{11}\omega_{20} + 3\lambda^2\omega_{21} + 3\nu^2\omega_{21} + \bar{\lambda}^2\omega_{30} + \nu^2\omega_{30}) \\
G_{22} &= -\,ad(4\nu^2\omega_{11}^2 + 2\lambda^2\omega_{12} + 2\nu^2\omega_{12} + 2\nu^2\omega_{02}\omega_{20} + 2\bar{\lambda}^2\omega_{21} + 2\nu^2\omega_{21})
\end{aligned}
$$

$$
\begin{aligned}
G_{50} &= -ad(20\nu^2\omega_{20}\omega_{30} + 5\lambda^2\omega_{40} + 5\nu^2\omega_{40}) \\
G_{41} &= -ad(12\nu^2\omega_{20}\omega_{21} + 8\nu^2\omega_{11}\omega_{30} + 4\lambda^2\omega_{31} + 4\nu^2\omega_{31} + \bar{\lambda}^2\omega_{40} + \nu^2\omega_{40}) \\
G_{32} &= -ad\big(3\nu^2\omega_{12}\omega_{20} + 2\nu^2\omega_{31} + 3\nu^2(\omega_{12}\omega_{20} + 2\omega_{11}\omega_{21} + \omega_{22}) \\
&\qquad + 3\lambda^2\omega_{22} + 2\nu^2\omega_{02}\omega_{30} + 2\bar{\lambda}^2\omega_{31} + 6\nu^2\omega_{11}\omega_{21}\big)
\end{aligned}
$$

10.6 The coefficients of $h(z)$

$$h_{20} = \frac{G_{20}}{(\lambda^2 - \lambda)}, \qquad h_{11} = \frac{G_{11}}{(\lambda\bar{\lambda} - \lambda)}, \qquad h_{02} = \frac{G_{02}}{(\bar{\lambda}^2 - \lambda)},$$

$$
\begin{aligned}
h_{30} = \frac{1}{\lambda^3 - \lambda}\Big(& 3G_{20}h_{20} + 3G_{11}\bar{h}_{02} + G_{30} - 3h_{20}\lambda(\lambda h_{20} + G_{20}) \\
& - 3h_{11}\lambda(\bar{\lambda}\bar{h}_{02} + \bar{G}_{02}) + 3h_{20}^2\lambda^3 + 3h_{11}\bar{h}_{02}\lambda^3\Big)
\end{aligned}
$$

$$h_{12} = \frac{1}{\lambda\bar{\lambda}^2 - \lambda}\Big(G_{20}h_{02} + 2G_{02}\bar{h}_{11} + G_{11}\bar{h}_{20} + 2G_{11}h_{11} + G_{12} - h_{20}\lambda(\lambda h_{02} + G_{02})$$
$$- 2h_{02}\bar{\lambda}(\bar{\lambda}\bar{h}_{11} + \bar{G}_{11}) - h_{11}\big(\lambda(\bar{\lambda}\bar{h}_{20} + \bar{G}_{20}) - 2\bar{\lambda}h_{11}(\lambda h_{11} + G_{11})\big)$$
$$+ h_{02}h_{20}\bar{\lambda}^2\lambda + 2h_{11}^2\bar{\lambda}^2\lambda + 2h_{02}\bar{h}_{11}\bar{\lambda}^2\lambda + h_{11}\bar{h}_{20}\bar{\lambda}^2\lambda\Big)$$

$$h_{03} = \frac{1}{\bar{\lambda}^3 - \lambda}\Big(3G_{02}\bar{h}_{20} + 3G_{11}h_{02} + G_{03} - 3h_{02}\bar{\lambda}(\bar{\lambda}\bar{h}_{20} + \bar{G}_{20})$$
$$- 3h_{11}\bar{\lambda}(\lambda h_{02} + G_{02}) + 3h_{02}\bar{h}_{20}\bar{\lambda}^3 + 3h_{11}h_{02}\bar{\lambda}^3\Big)$$

10.7 The coefficients of $h_0^{-1}(z)$

$$\tilde{h}_{20} = -h_{20}, \qquad \tilde{h}_{11} = -h_{11}, \qquad \tilde{h}_{02} = -h_{02}$$

$$\tilde{h}_{30} = 3h_{20}^2 - h_{30} + 3h_{11}\bar{h}_{02}, \qquad \tilde{h}_{21} = 3h_{11}h_{20} + h_{02}\bar{h}_{02} + 2h_{11}\bar{h}_{11}$$

$$\tilde{h}_{12} = 2h_{11}^2 - h_{12} + h_{02}h_{20} + 2h_{02}\bar{h}_{11} + h_{11}\bar{h}_{20}, \qquad \tilde{h}_{03} = 3h_{02}h_{11} - h_{03} + 3h_{02}\bar{h}_{20}$$

$$\tilde{h}_{40} = -15h_{20}^3 + 10h_{20}h_{30} - 30h_{11}h_{20}\bar{h}_{02} - 3h_{02}\bar{h}_{02}^2 + 4h_{11}\bar{h}_{03} - 12h_{11}\bar{h}_{02}\bar{h}_{11}$$

$$\tilde{h}_{31} = -15h_{11}h_{20}^2 + 4h_{11}h_{30} - 12h_{11}^2\bar{h}_{02} + 3h_{12}\bar{h}_{02} - 6h_{02}h_{20}\bar{h}_{02} + h_{02}\bar{h}_{03}$$
$$- 12h_{11}h_{20}\bar{h}_{11} - 6h_{02}\bar{h}_{02}\bar{h}_{11} - 6h_{11}\bar{h}_{11}^2 + 3h_{11}\bar{h}_{12} - 3h_{11}\bar{h}_{02}\bar{h}_{20}$$

$$\tilde{h}_{22} = -12h_{11}^2h_{20} + 3h_{12}h_{20} - 3h_{02}h_{20}^2 + h_{02}h_{30} + h_{03}\bar{h}_{02} - 9h_{02}h_{11}\bar{h}_{02} - 12h_{11}^2\bar{h}_{11}$$
$$+ 4h_{12}\bar{h}_{11} - 6h_{02}h_{20}\bar{h}_{11} - 6h_{02}\bar{h}_{11}^2 + 2h_{02}\bar{h}_{12} - 3h_{11}h_{20}\bar{h}_{20} - 3h_{02}\bar{h}_{02}\bar{h}_{20} - 6h_{11}\bar{h}_{11}\bar{h}_{20}$$

$$\tilde{h}_{13} = -6h_{11}^3 + 6h_{11}h_{12} + h_{03}h_{20} - 9h_{02}h_{11}h_{20} - 3h_{02}^2\bar{h}_{02} + 3h_{03}\bar{h}_{11} - 18h_{02}h_{11}\bar{h}_{11}$$
$$- 6h_{11}^2\bar{h}_{20} + 3h_{12}\bar{h}_{20} - 3h_{02}h_{20}\bar{h}_{20} - 12h_{02}\bar{h}_{11}\bar{h}_{20} - 3h_{11}\bar{h}_{20}^2 + h_{11}\bar{h}_{30}$$

$$\tilde{h}_{04} = 4h_{03}h_{11} - 12h_{02}h_{11}^2 + 6h_{02}h_{12} - 3h_{02}^2h_{20} - 12h_{02}^2\bar{h}_{11}$$
$$+ 6h_{03}\bar{h}_{20} - 18h_{02}h_{11}\bar{h}_{20} - 15h_{02}\bar{h}_{20}^2 + 4h_{02}\bar{h}_{30}$$

$$\tilde{h}_{50} = -5\Big(- 21h_{20}^4 + 4\bar{h}_{03}h_{11}(\bar{h}_{11} + 3h_{20}) - 3\bar{h}_{02}^2(2\bar{h}_{11}h_{02} + \bar{h}_{20}h_{11} + 6h_{11}^2 - h_{12} + 3h_{02}h_{20})$$
$$+ \bar{h}_{02}\big(2\bar{h}_{03}h_{02} - 3h_{11}(4\bar{h}_{11}^2 - 2\bar{h}_{12} + 12\bar{h}_{11}h_{20} + 21h_{20}^2 - 4h_{30})\big) + 21h_{20}^2h_{30} - 2h_{30}^2\Big)$$

$$
\begin{aligned}
\tilde{h}_{41} =& -24\bar{h}_{11}\bar{h}_{12}h_{11} + 24\bar{h}_{11}^3h_{11} - \bar{h}_{02}^2(-9\bar{h}_{20}h_{02} + 3h_{03} - 45h_{02}h_{11}) + 60\bar{h}_{11}^2h_{11}h_{20} \\
&+ 90\bar{h}_{11}h_{11}h_{20}^2 - 2\bar{h}_{03}(4\bar{h}_{11}h_{02} + 2\bar{h}_{20}h_{11} + 10h_{11}^2 - 2h_{12} + 5h_{02}h_{20}) \\
&+ 105h_{11}h_{20}^3 - 30\bar{h}_{12}h_{11}h_{20} - 20\bar{h}_{11}h_{11}h_{30} - 60h_{11}h_{20}h_{30} \\
&+ \bar{h}_{02}\Big(36\bar{h}_{11}^2h_{02} - 12\bar{h}_{12}h_{02} + 12\bar{h}_{11}(3\bar{h}_{20}h_{11} + 10h_{11}^2 - 2h_{12} + 5h_{02}h_{20}) \\
&+ 5(6\bar{h}_{20}h_{11}h_{20} + 36h_{11}^2h_{20} - 6h_{12}h_{20} + 9h_{02}h_{20}^2 - 2h_{02}h_{30})\Big)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h}_{32} =& -9\bar{h}_{12}\bar{h}_{20}h_{11} + 24\bar{h}_{11}^3h_{02} + 9\bar{h}_{02}^2h_{02}^2 + 9\bar{h}_{02}\bar{h}_{20}^2h_{11} - 3\bar{h}_{02}\bar{h}_{30}h_{11} - 24\bar{h}_{12}h_{11}^2 \\
&+ 36\bar{h}_{02}\bar{h}_{20}h_{11}^2 + 60\bar{h}_{02}h_{11}^3 - \bar{h}_{03}(3\bar{h}_{20}h_{02} - h_{03} + 12h_{02}h_{11}) + 6\bar{h}_{12}h_{12} \\
&- 9\bar{h}_{02}\bar{h}_{20}h_{12} - 36\bar{h}_{02}h_{11}h_{12} - 12\bar{h}_{12}h_{02}h_{20} + 18\bar{h}_{02}\bar{h}_{20}h_{02}h_{20} - 6\bar{h}_{02}h_{03}h_{20} \\
&+ 90\bar{h}_{02}h_{02}h_{11}h_{20} + 15\bar{h}_{20}h_{11}h_{20}^2 + 90h_{11}^2h_{20}^2 - 15h_{12}h_{20}^2 + 15h_{02}h_{20}^3 \\
&+ 18\bar{h}_{11}^2(2\bar{h}_{20}h_{11} + 4h_{11}^2 - h_{12} + 2h_{02}h_{20}) - 4\bar{h}_{20}h_{11}h_{30} - 20h_{11}^2h_{30} + 4h_{12}h_{30} \\
&- 10h_{02}h_{20}h_{30} - \bar{h}_{11}\Big(18\bar{h}_{12}h_{02} - 9\bar{h}_{02}(4\bar{h}_{20}h_{02} - h_{03} + 12h_{02}h_{11}) \\
&- 36\bar{h}_{20}h_{11}h_{20} - 120h_{11}^2h_{20} + 24h_{12}h_{20} - 30h_{02}h_{20}^2 + 8h_{02}h_{30}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h}_{23} =& -4\bar{h}_{02}\bar{h}_{30}h_{02} + 15\bar{h}_{02}\bar{h}_{20}^2h_{02} - 3\bar{h}_{03}h_{02}^2 - 6\bar{h}_{02}\bar{h}_{20}h_{03} + 54\bar{h}_{02}\bar{h}_{20}h_{02}h_{11} \\
&- 12\bar{h}_{02}h_{03}h_{11} + 72\bar{h}_{02}h_{02}h_{11}^2 - 3\bar{h}_{12}(4\bar{h}_{20}h_{02} - h_{03} + 9h_{02}h_{11}) \\
&+ 12\bar{h}_{11}^2(5\bar{h}_{20}h_{02} - h_{03} + 9h_{02}h_{11}) - 18\bar{h}_{02}h_{02}h_{12} + 18\bar{h}_{02}h_{02}^2h_{20} \\
&+ 9\bar{h}_{20}^2h_{11}h_{20} - 3\bar{h}_{30}h_{11}h_{20} + 36\bar{h}_{20}h_{11}^2h_{20} + 60h_{11}^3h_{20} - 9\bar{h}_{20}h_{12}h_{20} \\
&- 36h_{11}h_{12}h_{20} + 9\bar{h}_{20}h_{02}h_{20}^2 - 3h_{03}h_{20}^2 + 45h_{02}h_{11}h_{20}^2 - 12h_{02}h_{11}h_{30} \\
&+ \bar{h}_{11}\Big(36\bar{h}_{02}h_{02}^2 + 30\bar{h}_{20}^2h_{11} - 8\bar{h}_{30}h_{11} + 72h_{11}^3 - 54h_{11}h_{12} - 9h_{03}h_{20} \\
&+ 108h_{02}h_{11}h_{20} + 12\bar{h}_{20}(6h_{11}^2 - 2h_{12} + 3h_{02}h_{20})\Big) - 3\bar{h}_{20}h_{02}h_{30} + h_{03}h_{30}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h}_{14} =& -12\bar{h}_{12}h_{02}^2 + 60\bar{h}_{11}^2h_{02}^2 + 30\bar{h}_{02}\bar{h}_{20}h_{02}^2 - 10\bar{h}_{02}h_{02}h_{03} + 15\bar{h}_{20}^3h_{11} - 10\bar{h}_{20}\bar{h}_{30}h_{11} \\
&+ 45\bar{h}_{02}h_{02}^2h_{11} + 30\bar{h}_{20}^2h_{11}^2 - 8\bar{h}_{30}h_{11}^2 + 36\bar{h}_{20}h_{11}^3 + 24h_{11}^4 - 15\bar{h}_{20}^2h_{12} \\
&+ 4\bar{h}_{30}h_{12} - 36\bar{h}_{20}h_{11}h_{12} - 36h_{11}^2h_{12} + 6h_{12}^2 + 15\bar{h}_{20}^2h_{02}h_{20} - 4\bar{h}_{30}h_{02}h_{20} \\
&- 6\bar{h}_{20}h_{03}h_{20} + 54\bar{h}_{20}h_{02}h_{11}h_{20} - 12h_{03}h_{11}h_{20} + 72h_{02}h_{11}^2h_{20} \\
&- 18h_{02}h_{12}h_{20} + 9h_{02}^2h_{20}^2 + 2\bar{h}_{11}\Big(45\bar{h}_{20}^2h_{02} - 15\bar{h}_{20}(h_{03} - 6h_{02}h_{11}) \\
&- 2\big(5\bar{h}_{30}h_{02} + 8h_{03}h_{11} - 3h_{02}(12h_{11}^2 - 4h_{12} + 3h_{02}h_{20})\big)\Big) + 3h_{02}^2h_{30}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h}_{05} = -5\Big(& -21\bar{h}_{20}^3h_{02} - 3\bar{h}_{02}h_{02}^3 - 2\bar{h}_{30}h_{03} + 10\bar{h}_{11}h_{02}h_{03} + 6\bar{h}_{30}h_{02}h_{11} \\
&- 30\bar{h}_{11}h_{02}^2h_{11} + 4h_{03}h_{11}^2 - 12h_{02}h_{11}^3 + 9\bar{h}_{20}^2(h_{03} - 3h_{02}h_{11}) \\
&- 2h_{03}h_{12} + 12h_{02}h_{11}h_{12} + 2h_{02}h_{03}h_{20} - 9h_{02}^2h_{11}h_{20} \\
&+ 2\bar{h}_{20}(6\bar{h}_{30}h_{02} - 18\bar{h}_{11}h_{02}^2 + 4h_{03}h_{11} - 12h_{02}h_{11}^2 + 6h_{02}h_{12} - 3h_{02}^2h_{20})\Big)
\end{aligned}
$$

10.8 The lower order terms of $G(h(w))$

The composition of $G(z)$ and $h(w)$ can be written in the following form

$$G(h(w)) = \sum_{1\le i+j\le 5} \frac{\alpha_{ij}}{i!j!} w^i \overline{w}^j + R_5,$$

where $R_5 = R_5(a, w, \overline{w}, c) = O(|w|^6)$ and $\alpha_{ij} = \alpha_{ij}(a)$ is complex.

$$\alpha_{10} = \lambda \qquad \alpha_{20} = G_{20} + h_{20}\lambda$$
$$\alpha_{11} = G_{11} + h_{11}\lambda \qquad \alpha_{02} = G_{02} + h_{02}\lambda$$

$$\begin{aligned}
\alpha_{30} &= G_{30} + 3G_{20}h_{20} + h_{30}\lambda + 3G_{11}\bar{h}_{02} \\
\alpha_{21} &= G_{21} + 2G_{20}h_{11} + G_{11}h_{20} + G_{02}\bar{h}_{02} + 2G_{11}\bar{h}_{11} \\
\alpha_{12} &= G_{12} + G_{20}h_{02} + 2G_{11}h_{11} + h_{12}\lambda + 2G_{02}\bar{h}_{11} + G_{11}\bar{h}_{20} \\
\alpha_{03} &= G_{03} + 3G_{11}h_{02} + h_{03}\lambda + 3G_{02}\bar{h}_{20}
\end{aligned}$$

$$\alpha_{40} = G_{40} + 6G_{30}h_{20} + 3G_{20}h_{20}^2 + 4G_{20}h_{30} + 6(G_{21} + G_{11}h_{20})\bar{h}_{02} + 3G_{02}\bar{h}_{02}^2 + 4G_{11}\bar{h}_{03}$$

$$\begin{aligned}
\alpha_{31} = {} & G_{31} + 3G_{30}h_{11} + 3G_{21}h_{20} + 3G_{20}h_{11}h_{20} + G_{11}h_{30} + G_{02}\bar{h}_{03} + 3G_{21}\bar{h}_{11} \\
& + 3G_{11}h_{20}\bar{h}_{11} + 3\bar{h}_{02}(G_{12} + G_{11}h_{11} + G_{02}\bar{h}_{11}) + 3G_{11}\bar{h}_{12}
\end{aligned}$$

$$\begin{aligned}
\alpha_{22} = {} & G_{22} + G_{30}h_{02} + 4G_{21}h_{11} + 2G_{20}h_{11}^2 + 2G_{20}h_{12} + G_{12}h_{20} + G_{20}h_{02}h_{20} \\
& + 4(G_{12} + G_{11}h_{11})\bar{h}_{11} + 2G_{02}\bar{h}_{11}^2 + 2G_{02}\bar{h}_{12} + G_{21}\bar{h}_{20} + G_{11}h_{20}\bar{h}_{20} \\
& + \bar{h}_{02}(G_{03} + G_{11}h_{02} + G_{02}\bar{h}_{20})
\end{aligned}$$

$$\begin{aligned}
\alpha_{13} = {} & G_{13} + 3G_{21}h_{02} + G_{20}h_{03} + 3G_{12}h_{11} + 3G_{20}h_{02}h_{11} + 3G_{11}h_{12} \\
& + 3(G_{12} + G_{11}h_{11})\bar{h}_{20} + 3\bar{h}_{11}(G_{03} + G_{11}h_{02} + G_{02}\bar{h}_{20}) + G_{11}\bar{h}_{30};
\end{aligned}$$

$$\alpha_{04} = G_{04} + 6G_{12}h_{02} + 3G_{20}h_{02}^2 + 4G_{11}h_{03} + 6(G_{03} + G_{11}h_{02})\bar{h}_{20} + 3G_{02}\bar{h}_{20}^2 + 4G_{02}\bar{h}_{30}$$

$$\begin{aligned}
\alpha_{50} = {} & G_{50} + 10G_{40}h_{20} + 15G_{30}h_{20}^2 + 10G_{30}h_{30} + 10G_{20}h_{20}h_{30} + 15G_{12}\bar{h}_{02}^2 \\
& + 10(G_{21} + G_{11}h_{20})\bar{h}_{03} + 10\bar{h}_{02}(G_{31} + 3G_{21}h_{20} + G_{11}h_{30} + G_{02}\bar{h}_{03})
\end{aligned}$$

$$\begin{aligned}
\alpha_{41} = {} & G_{41} + 4G_{40}h_{11} + 6G_{31}h_{20} + 12G_{30}h_{11}h_{20} + 3G_{21}h_{20}^2 + 4G_{21}h_{30} \\
& + 4G_{20}h_{11}h_{30} + 3G_{03}\bar{h}_{02}^2 + 4G_{31}\bar{h}_{11} + 12G_{21}h_{20}\bar{h}_{11} + 4G_{11}h_{30}\bar{h}_{11} \\
& + 4\bar{h}_{03}(G_{12} + G_{11}h_{11} + G_{02}\bar{h}_{11}) + 6G_{21}\bar{h}_{12} + 6G_{11}h_{20}\bar{h}_{12} \\
& + 6\bar{h}_{02}(G_{22} + 2G_{21}h_{11} + G_{12}h_{20} + 2G_{12}\bar{h}_{11} + G_{02}\bar{h}_{12})
\end{aligned}$$

$$\begin{aligned}\alpha_{32} = {} & G_{32} + G_{40}h_{02} + 6G_{31}h_{11} + 6G_{30}h_{11}^2 + 3G_{30}h_{12} + 3G_{22}h_{20} + 3G_{30}h_{02}h_{20} \\ & + 6G_{21}h_{11}h_{20} + 3G_{20}h_{12}h_{20} + G_{12}h_{30} + G_{20}h_{02}h_{30} + 6G_{22}\bar{h}_{11} + 12G_{21}h_{11}\bar{h}_{11} \\ & + 6G_{12}h_{20}\bar{h}_{11} + 6G_{12}\bar{h}_{11}^2 + 6G_{12}\bar{h}_{12} + 6G_{11}h_{11}\bar{h}_{12} + 6G_{02}\bar{h}_{11}\bar{h}_{12} \\ & + G_{31}\bar{h}_{20} + 3G_{21}h_{20}\bar{h}_{20} + G_{11}h_{30}\bar{h}_{20} + \bar{h}_{03}(G_{03} + G_{11}h_{02} + G_{02}\bar{h}_{20}) \\ & + 3\bar{h}_{02}(G_{13} + G_{21}h_{02} + 2G_{12}h_{11} + G_{11}h_{12} + 2G_{03}\bar{h}_{11} + G_{12}\bar{h}_{20})\end{aligned}$$

$$\begin{aligned}\alpha_{23} = {} & G_{23} + 3G_{31}h_{02} + G_{30}h_{03} + 6G_{22}h_{11} + 6G_{30}h_{02}h_{11} + 6G_{21}h_{11}^2 + 6G_{21}h_{12} \\ & + 6G_{20}h_{11}h_{12} + G_{13}h_{20} + 3G_{21}h_{02}h_{20} + G_{20}h_{03}h_{20} + 6G_{03}\bar{h}_{11}^2 + 3G_{03}\bar{h}_{12} \\ & + 3G_{11}h_{02}\bar{h}_{12} + 3G_{22}\bar{h}_{20} + 6G_{21}h_{11}\bar{h}_{20} + 3G_{12}h_{20}\bar{h}_{20} + 3G_{02}\bar{h}_{12}\bar{h}_{20} \\ & + 6\bar{h}_{11}(G_{13} + G_{21}h_{02} + 2G_{12}h_{11} + G_{11}h_{12} + G_{12}\bar{h}_{20}) + G_{21}\bar{h}_{30} \\ & + G_{11}h_{20}\bar{h}_{30} + \bar{h}_{02}(G_{04} + 3G_{12}h_{02} + G_{11}h_{03} + 3G_{03}\bar{h}_{20} + G_{02}\bar{h}_{30})\end{aligned}$$

$$\begin{aligned}\alpha_{14} = {} & G_{14} + 6G_{22}h_{02} + 3G_{30}h_{02}^2 + 4G_{21}h_{03} + 4G_{13}h_{11} + 12G_{21}h_{02}h_{11} + 4G_{20}h_{03}h_{11} \\ & + 6G_{12}h_{12} + 6G_{20}h_{02}h_{12} + 6(G_{13} + G_{21}h_{02} + 2G_{12}h_{11} + G_{11}h_{12})\bar{h}_{20} + 3G_{12}\bar{h}_{20}^2 \\ & + 4G_{12}\bar{h}_{30} + 4G_{11}h_{11}\bar{h}_{30} + 4\bar{h}_{11}(G_{04} + 3G_{12}h_{02} + G_{11}h_{03} + 3G_{03}\bar{h}_{20} + G_{02}\bar{h}_{30})\end{aligned}$$

$$\begin{aligned}\alpha_{05} = {} & G_{05} + 10G_{13}h_{02} + 15G_{21}h_{02}^2 + 10G_{12}h_{03} + 10G_{20}h_{02}h_{03} + 15G_{03}\bar{h}_{20}^2 \\ & + 10(G_{03} + G_{11}h_{02})\bar{h}_{30} + 10\bar{h}_{20}(G_{04} + 3G_{12}h_{02} + G_{11}h_{03} + G_{02}\bar{h}_{30})\end{aligned}$$

10.9 The lower order terms of $h^{-1}(G(h(w)))$

$$\begin{aligned}\beta_{40} = {} & \alpha_{40} + 3\alpha_{20}^2\tilde{h}_{20} + 4\alpha_{10}\alpha_{30}\tilde{h}_{20} + 6\alpha_{10}^2\alpha_{20}\tilde{h}_{30} + \alpha_{10}^4\tilde{h}_{40} \\ & + 6(\alpha_{20}\tilde{h}_{11} + \alpha_{10}^2\tilde{h}_{21})\bar{\alpha}_{02} + 3\tilde{h}_{02}\bar{\alpha}_{02}^2 + 4\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{03}\end{aligned}$$

$$\begin{aligned}\beta_{31} = {} & \alpha_{31} + 3\alpha_{11}\alpha_{20}\tilde{h}_{20} + 3\alpha_{10}\alpha_{21}\tilde{h}_{20} + 3\alpha_{10}^2\alpha_{11}\tilde{h}_{30} \\ & + (\alpha_{30}\tilde{h}_{11} + 3\alpha_{10}\alpha_{20}\tilde{h}_{21} + \alpha_{10}^3\tilde{h}_{31} + \tilde{h}_{02}\bar{\alpha}_{03})\bar{\alpha}_{10} + 3\alpha_{20}\tilde{h}_{11}\bar{\alpha}_{11} \\ & + 3\alpha_{10}^2\tilde{h}_{21}\bar{\alpha}_{11} + 3\bar{\alpha}_{02}(\alpha_{11}\tilde{h}_{11} + \alpha_{10}\tilde{h}_{12}\bar{\alpha}_{10} + \tilde{h}_{02}\bar{\alpha}_{11}) + 3\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{12}\end{aligned}$$

$$\begin{aligned}\beta_{22} = {} & \alpha_{22} + 2\alpha_{11}^2\tilde{h}_{20} + 2\alpha_{10}\alpha_{12}\tilde{h}_{20} + \alpha_{02}\alpha_{20}\tilde{h}_{20} + \alpha_{02}\alpha_{10}^2\tilde{h}_{30} + (\alpha_{20}\tilde{h}_{12} + \alpha_{10}^2\tilde{h}_{22})\bar{\alpha}_{10}^2 \\ & + 4\alpha_{11}\tilde{h}_{11}\bar{\alpha}_{11} + 2\tilde{h}_{02}\bar{\alpha}_{11}^2 + 2\bar{\alpha}_{10}(\alpha_{21}\tilde{h}_{11} + 2\alpha_{10}\alpha_{11}\tilde{h}_{21} + 2\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{11} + \tilde{h}_{02}\bar{\alpha}_{12}) \\ & + \alpha_{20}\tilde{h}_{11}\bar{\alpha}_{20} + \alpha_{10}^2\tilde{h}_{21}\bar{\alpha}_{20} + \bar{\alpha}_{02}(\alpha_{02}\tilde{h}_{11} + \tilde{h}_{03}\bar{\alpha}_{10}^2 + \tilde{h}_{02}\bar{\alpha}_{20}) + 2\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{21}\end{aligned}$$

$$\begin{aligned}\beta_{13} = {} & \alpha_{13} + \alpha_{03}\alpha_{10}\tilde{h}_{20} + 3\alpha_{02}\alpha_{11}\tilde{h}_{20} + \alpha_{10}\tilde{h}_{13}\bar{\alpha}_{10}^3 + 3\bar{\alpha}_{10}^2(\alpha_{11}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{11}) \\ & + 3\alpha_{11}\tilde{h}_{11}\bar{\alpha}_{20} + 3\bar{\alpha}_{11}(\alpha_{02}\tilde{h}_{11} + \tilde{h}_{02}\bar{\alpha}_{20}) + \alpha_{10}\tilde{h}_{11}\bar{\alpha}_{30} \\ & + 3\bar{\alpha}_{10}(\alpha_{12}\tilde{h}_{11} + \alpha_{02}\alpha_{10}\tilde{h}_{21} + \alpha_{10}\tilde{h}_{12}\bar{\alpha}_{20} + \tilde{h}_{02}\bar{\alpha}_{21})\end{aligned}$$

$$
\begin{aligned}
\beta_{04} = {} & \alpha_{04} + 3\alpha_{02}^2\tilde{h}_{20} + \tilde{h}_{04}\bar{\alpha}_{10}^4 + 6\alpha_{02}\tilde{h}_{11}\bar{\alpha}_{20} + 3\tilde{h}_{02}\bar{\alpha}_{20}^2 \\
& + 6\bar{\alpha}_{10}^2(\alpha_{02}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{20}) + 4\bar{\alpha}_{10}(\alpha_{03}\tilde{h}_{11} + \tilde{h}_{02}\bar{\alpha}_{30})
\end{aligned}
$$

$$
\begin{aligned}
\beta_{50} = {} & \alpha_{50} + 10\alpha_{20}\alpha_{30}\tilde{h}_{20} + 5\alpha_{10}\alpha_{40}\tilde{h}_{20} + 15\alpha_{10}\alpha_{20}^2\tilde{h}_{30} + 10\alpha_{10}^2\alpha_{30}\tilde{h}_{30} + 10\alpha_{10}^3\alpha_{20}\tilde{h}_{40} \\
& + \alpha_{10}^5\tilde{h}_{50} + 15\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{02}^2 + 10(\alpha_{20}\tilde{h}_{11} + \alpha_{10}^2\tilde{h}_{21})\bar{\alpha}_{03} \\
& + 10\bar{\alpha}_{02}(\alpha_{30}\tilde{h}_{11} + 3\alpha_{10}\alpha_{20}\tilde{h}_{21} + \alpha_{10}^3\tilde{h}_{31} + \tilde{h}_{02}\bar{\alpha}_{03}) + 5\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{04}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{41} = {} & \alpha_{41} + 6\alpha_{20}\alpha_{21}\tilde{h}_{20} + 4\alpha_{11}\alpha_{30}\tilde{h}_{20} + 4\alpha_{10}\alpha_{31}\tilde{h}_{20} + 12\alpha_{10}\alpha_{11}\alpha_{20}\tilde{h}_{30} + 6\alpha_{10}^2\alpha_{21}\tilde{h}_{30} \\
& + 4\alpha_{10}^3\alpha_{11}\tilde{h}_{40} + \alpha_{40}\tilde{h}_{11}\bar{\alpha}_{10} + 3\alpha_{20}^2\tilde{h}_{21}\bar{\alpha}_{10} + 4\alpha_{10}\alpha_{30}\tilde{h}_{21}\bar{\alpha}_{10} + 6\alpha_{10}^2\alpha_{20}\tilde{h}_{31}\bar{\alpha}_{10} \\
& + \alpha_{10}^4\tilde{h}_{41}\bar{\alpha}_{10} + 3\tilde{h}_{03}\bar{\alpha}_{02}^2\bar{\alpha}_{10} + \tilde{h}_{02}\bar{\alpha}_{04}\bar{\alpha}_{10} + 4\alpha_{30}\tilde{h}_{11}\bar{\alpha}_{11} + 12\alpha_{10}\alpha_{20}\tilde{h}_{21}\bar{\alpha}_{11} \\
& + 4\alpha_{10}^3\tilde{h}_{31}\bar{\alpha}_{11} + 4\bar{\alpha}_{03}(\alpha_{11}\tilde{h}_{11} + \alpha_{10}\tilde{h}_{12}\bar{\alpha}_{10} + \tilde{h}_{02}\bar{\alpha}_{11}) + 6\alpha_{20}\tilde{h}_{11}\bar{\alpha}_{12} \\
& + 6\bar{\alpha}_{02}\Big(\alpha_{21}\tilde{h}_{11} + 2\alpha_{10}\alpha_{11}\tilde{h}_{21} + (\alpha_{20}\tilde{h}_{12} + \alpha_{10}^2\tilde{h}_{22})\bar{\alpha}_{10} + 2\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{11} + \tilde{h}_{02}\bar{\alpha}_{12}\Big) \\
& + 6\alpha_{10}^2\tilde{h}_{21}\bar{\alpha}_{12} + 4\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{13}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{32} = {} & \alpha_{32} + 3\alpha_{12}\alpha_{20}\tilde{h}_{20} + 6\alpha_{11}\alpha_{21}\tilde{h}_{20} + 3\alpha_{10}\alpha_{22}\tilde{h}_{20} + \alpha_{02}\alpha_{30}\tilde{h}_{20} + 6\alpha_{10}\alpha_{11}^2\tilde{h}_{30} \\
& + 3\alpha_{10}^2\alpha_{12}\tilde{h}_{30} + 3\alpha_{02}\alpha_{10}\alpha_{20}\tilde{h}_{30} + \alpha_{02}\alpha_{10}^3\tilde{h}_{40} + 2\alpha_{31}\tilde{h}_{11}\bar{\alpha}_{10} + 6\alpha_{11}\alpha_{20}\tilde{h}_{21}\bar{\alpha}_{10} \\
& + 6\alpha_{10}\alpha_{21}\tilde{h}_{21}\bar{\alpha}_{10} + 6\alpha_{10}^2\alpha_{11}\tilde{h}_{31}\bar{\alpha}_{10} + \alpha_{30}\tilde{h}_{12}\bar{\alpha}_{10}^2 + 3\alpha_{10}\alpha_{20}\tilde{h}_{22}\bar{\alpha}_{10}^2 \\
& + \alpha_{10}^3\tilde{h}_{32}\bar{\alpha}_{10}^2 + 6\alpha_{21}\tilde{h}_{11}\bar{\alpha}_{11} + 12\alpha_{10}\alpha_{11}\tilde{h}_{21}\bar{\alpha}_{11} + 6\alpha_{20}\tilde{h}_{12}\bar{\alpha}_{10}\bar{\alpha}_{11} \\
& + 6\alpha_{10}^2\tilde{h}_{22}\bar{\alpha}_{10}\bar{\alpha}_{11} + 6\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{11}^2 + 6\alpha_{11}\tilde{h}_{11}\bar{\alpha}_{12} + 6\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{10}\bar{\alpha}_{12} \\
& + 6\tilde{h}_{02}\bar{\alpha}_{11}\bar{\alpha}_{12} + 2\tilde{h}_{02}\bar{\alpha}_{10}\bar{\alpha}_{13} + \alpha_{30}\tilde{h}_{11}\bar{\alpha}_{20} + 3\alpha_{10}\alpha_{20}\tilde{h}_{21}\bar{\alpha}_{20} + \alpha_{10}^3\tilde{h}_{31}\bar{\alpha}_{20} \\
& + \bar{\alpha}_{03}(\alpha_{02}\tilde{h}_{11} + \tilde{h}_{03}\bar{\alpha}_{10}^2 + \tilde{h}_{02}\bar{\alpha}_{20}) + 3\alpha_{20}\tilde{h}_{11}\bar{\alpha}_{21} + 3\alpha_{10}^2\tilde{h}_{21}\bar{\alpha}_{21} \\
& + 3\bar{\alpha}_{02}\Big(\alpha_{12}\tilde{h}_{11} + \alpha_{02}\alpha_{10}\tilde{h}_{21} + \alpha_{10}\tilde{h}_{13}\bar{\alpha}_{10}^2 + 2\bar{\alpha}_{10}(\alpha_{11}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{11}) \\
& + \alpha_{10}\tilde{h}_{12}\bar{\alpha}_{20} + \tilde{h}_{02}\bar{\alpha}_{21}\Big) + 3\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{22}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{23} = {} & \alpha_{23} + 6\alpha_{11}\alpha_{12}\tilde{h}_{20} + 2\alpha_{10}\alpha_{13}\tilde{h}_{20} + \alpha_{03}\alpha_{20}\tilde{h}_{20} + 3\alpha_{02}\alpha_{21}\tilde{h}_{20} + \alpha_{03}\alpha_{10}^2\tilde{h}_{30} \\
& + 6\alpha_{02}\alpha_{10}\alpha_{11}\tilde{h}_{30} + (\alpha_{20}\tilde{h}_{13} + \alpha_{10}^2\tilde{h}_{23})\bar{\alpha}_{10}^3 + 6\alpha_{12}\tilde{h}_{11}\bar{\alpha}_{11} + 6\alpha_{02}\alpha_{10}\tilde{h}_{21}\bar{\alpha}_{11} \\
& + 3\alpha_{02}\tilde{h}_{11}\bar{\alpha}_{12} + 3\bar{\alpha}_{10}^2(\alpha_{21}\tilde{h}_{12} + 2\alpha_{10}\alpha_{11}\tilde{h}_{22} + 2\alpha_{10}\tilde{h}_{13}\bar{\alpha}_{11} + \tilde{h}_{03}\bar{\alpha}_{12}) + 3\alpha_{21}\tilde{h}_{11}\bar{\alpha}_{20} \\
& + 6\alpha_{10}\alpha_{11}\tilde{h}_{21}\bar{\alpha}_{20} + 6\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{11}\bar{\alpha}_{20} + 3\tilde{h}_{02}\bar{\alpha}_{12}\bar{\alpha}_{20} + 6\alpha_{11}\tilde{h}_{11}\bar{\alpha}_{21} \\
& + 6\tilde{h}_{02}\bar{\alpha}_{11}\bar{\alpha}_{21} + 3\bar{\alpha}_{10}(\alpha_{22}\tilde{h}_{11} + 2\alpha_{11}^2\tilde{h}_{21} + 2\alpha_{10}\alpha_{12}\tilde{h}_{21} + \alpha_{02}\alpha_{20}\tilde{h}_{21} + \alpha_{02}\alpha_{10}^2\tilde{h}_{31} \\
& + 4\alpha_{11}\tilde{h}_{12}\bar{\alpha}_{11} + 2\tilde{h}_{03}\bar{\alpha}_{11}^2 + \alpha_{20}\tilde{h}_{12}\bar{\alpha}_{20} + \alpha_{10}^2\tilde{h}_{22}\bar{\alpha}_{20} + 2\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{21} + \tilde{h}_{02}\bar{\alpha}_{22}) \\
& + \alpha_{20}\tilde{h}_{11}\bar{\alpha}_{30} + 2\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{31} + \alpha_{10}^2\tilde{h}_{21}\bar{\alpha}_{30} \\
& + \bar{\alpha}_{02}\big(\alpha_{03}\tilde{h}_{11} + \tilde{h}_{04}\bar{\alpha}_{10}^3 + 3\bar{\alpha}_{10}(\alpha_{02}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{20}) + \tilde{h}_{02}\bar{\alpha}_{30}\big)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{14} = {} & \alpha_{14} + \alpha_{04}\alpha_{10}\tilde{h}_{20} + 4\alpha_{03}\alpha_{11}\tilde{h}_{20} + 6\alpha_{02}\alpha_{12}\tilde{h}_{20} + 3\alpha_{02}^2\alpha_{10}\tilde{h}_{30} + \alpha_{10}\tilde{h}_{14}\bar{\alpha}_{10}^4 \\
& + 4\bar{\alpha}_{10}^3(\alpha_{11}\tilde{h}_{13} + \tilde{h}_{04}\bar{\alpha}_{11}) + 6\alpha_{12}\tilde{h}_{11}\bar{\alpha}_{20} + 6\alpha_{02}\alpha_{10}\tilde{h}_{21}\bar{\alpha}_{20} + 3\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{20}^2 \\
& + 6\alpha_{02}\tilde{h}_{11}\bar{\alpha}_{21} + 6\tilde{h}_{02}\bar{\alpha}_{20}\bar{\alpha}_{21} + 6\bar{\alpha}_{10}^2(\alpha_{12}\tilde{h}_{12} + \alpha_{02}\alpha_{10}\tilde{h}_{22} + \alpha_{10}\tilde{h}_{13}\bar{\alpha}_{20} + \tilde{h}_{03}\bar{\alpha}_{21}) \\
& + 4\alpha_{11}\tilde{h}_{11}\bar{\alpha}_{30} + 4\bar{\alpha}_{11}(\alpha_{03}\tilde{h}_{11} + \tilde{h}_{02}\bar{\alpha}_{30}) + 4\bar{\alpha}_{10}\big(\alpha_{13}\tilde{h}_{11} + \alpha_{03}\alpha_{10}\tilde{h}_{21} + 3\alpha_{02}\alpha_{11}\tilde{h}_{21} \\
& + 3\alpha_{11}\tilde{h}_{12}\bar{\alpha}_{20} + 3\bar{\alpha}_{11}(\alpha_{02}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{20}) + \alpha_{10}\tilde{h}_{12}\bar{\alpha}_{30} + \tilde{h}_{02}\bar{\alpha}_{31}\big) + \alpha_{10}\tilde{h}_{11}\bar{\alpha}_{40}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{05} = {} & \alpha_{05} + 10\alpha_{02}\alpha_{03}\tilde{h}_{20} + \tilde{h}_{05}\bar{\alpha}_{10}^5 + 10\bar{\alpha}_{10}^3(\alpha_{02}\tilde{h}_{13} + \tilde{h}_{04}\bar{\alpha}_{20}) + 10\alpha_{02}\tilde{h}_{11}\bar{\alpha}_{30} \\
& + 10\bar{\alpha}_{20}(\alpha_{03}\tilde{h}_{11} + \tilde{h}_{02}\bar{\alpha}_{30}) + 10\bar{\alpha}_{10}^2(\alpha_{03}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{30}) \\
& + 5\bar{\alpha}_{10}(\alpha_{04}\tilde{h}_{11} + 3\alpha_{02}^2\tilde{h}_{21} + 6\alpha_{02}\tilde{h}_{12}\bar{\alpha}_{20} + 3\tilde{h}_{03}\bar{\alpha}_{20}^2 + \tilde{h}_{02}\bar{\alpha}_{40})
\end{aligned}
$$

10.10 The Lyapunov-coefficient

$$
c_1 = \frac{G_{11}G_{20}(2\lambda + \bar{\lambda} - 3)}{2(\lambda^2 - \lambda)(\bar{\lambda} - 1)} + \frac{|G_{11}|^2}{|\lambda|^2 - \bar{\lambda}} + \frac{|G_{02}|^2}{2(\lambda^2 - \bar{\lambda})} + \frac{G_{21}}{2}
$$

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